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Methods in M-theory

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In these notes I discuss various aspects of the elusive M-theory, with a special stress on the structure of the supergravity effective descriptions and their relations.

Updated version at http://cern.ch/pvanhove/papiers.html

Contents

1. Introducing the supergravities

Supersymmetric string theories are consistently defined in ten dimensions. This is a lower bound on the dimension where this quantum theory can be defined. There could be extra dimensions invisible in the string perturbation scheme which open up when considering strong coupling string effects. On this other hand, eleven is the maximal dimension for a supergravity theory [1].[1] The supergravity in eleven dimensions is so constrainted by the supersymmetries and gauge invariance that the Lagrangian written by Cremmer, Julia & Scherk (CJS) [2] is unique (in particular a cosmological constant term is even not allowed [3]). So we only have the possibility for ten- or eleven-dimensional spacetimes. It is by now clearly established that the low-energy limit —or infinite tension limit— of superstrings models are anomaly free supergravity theories [4,5]. There are five superstring theories in ten dimensions, namely, (i) the closed non-chiral superstring theory of type IIa, (ii) the closed chiral superstring theory of type IIb, (iii) the heterotic superstring with gauge group $E_8 \times E_8$, (iv) the heterotic superstring with gauge group $SO(32)$ and the (v) open superstring theory with gauge group $SO(32)$ [6]. Their associated supergravity theories are, respectively, (i) the $\mathcal{N}_4 = 8$ non-chiral supergravity of type IIa, (ii) the $\mathcal{N}_4 = 8$ chiral supergravity of type IIb, (iii) the $\mathcal{N}_4 = 4$ supergravity with super Yang–Mills gauge group $E_8 \times E_8$, and (iv) & (v) the $\mathcal{N}_4 = 4$ supergravity with super Yang–Mills gauge group $SO(32)$. We already see that the two different superstring theories (iv) and (v) have the same effective supergravity theory. Since their string perturbative spectrum is really different this equivalence must involve the non-perturbative sector.

It is now understood that the dilaton, which serves to defines the asymptotic perturbative string expansion $g_s = \exp\langle\phi\rangle$, plays a central role in probing various regimes of the superstring theories. First, it is a special case of the numerous scalar fields that parametrize the vacua of the supergravities theories. Second, it is known that reinterpreting these fields lead to equivalence of between the supergravity theories and then between the string theories (see [7] for an example). Townsend [8] and Witten [9] explained that the expectation value of the dilaton is fixed by the size of an eleventh radius of compactification, $R_{11} = g_s l_s$. Such that the strong coupling string limit $g_s \to \infty$ is a eleven dimensional spacetime. This is satisfactory since we have now a unique (effective) theory in eleven dimensions, related to the other theories by a Kaluza–Klein compactification. Fortunatly, the no-go theorem stating that no chiral theories can be obtained by Kaluza–Klein reduction of non-chiral theories [10], is circunvented if the extended objects of Mtwo-brane and the Mfive-brane are considered. This makes use the massive modes not present in the Kaluza–Klein theories. The compactification of the eleven-dimension supergravity theory down to nine-dimension

[1] Such theories defined with the usual Minkowsky metric, in dimensions higher than eleven, have massless particles of spin bigger than 2 and more than one graviton when reduced down to four dimensions. Particles with spin bigger than 2 are difficult to couple consistently to the gravity.

on a two-torus T^2, is a nine dimensional supergravity theory $\mathcal{N}_4 = 8$ coupled to a Kaluza–Klein tower of massive spin 2 multiplets. In the limit in which the area of the torus goes to zero, at fixed shape, one obtains the non-chiral D=9 supergravity theory, but in the limit of vanishing volume with fixed complex structure, $R_{11}R_{10} \to 0$ and R_{10}/R_{11} hold fixed, one obtains the ten dimension chiral type IIb supergavity [11,12]. Since we obtained a chiral theory there are two equivalent versions of this theory depending on the chirality of the fermions with respect to the Spin(9) group. This choice depends on the sign of the Chern-Simon coupling in D=11 effective Lagrangian[2] or equivalently to the sign of the Wess-Zumino term in the action of the Mtwo-brane. Already at this level the CJS supergravity knows about the membrane, and so by compactify about strings.

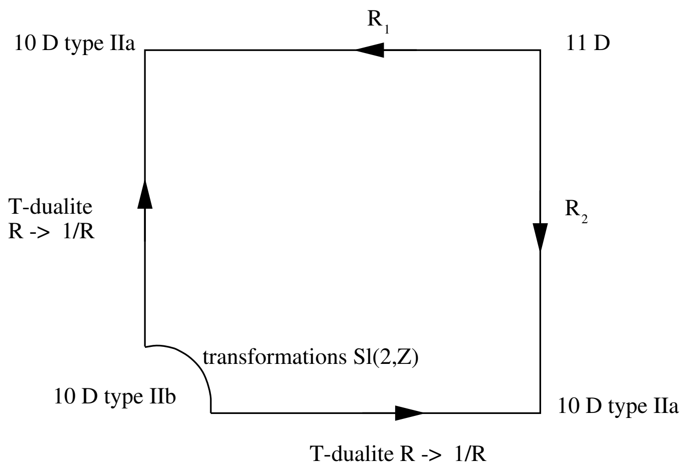

Fig. 1: Diagram showing the connection between the eleven-dimensional supergravity, the ten-dimensional non-chiral of type IIa theory and chiral type IIb theory.

For supergravity theories the distinction between string perturbative and non-perturbative states disappear since the parameter of the expansion is one of the many scalar field that parametrize the theory and has nothing to do with a perturbative expansion. Any extended object couple to massless fields through Wess-Zumino like interactions and their charges are constrainted by Dirac quantization relations. Moreover, these objects have to be well-defined classical solutions of the supergravity theories and already appear as (generalized) central charges of the super-algebra (see the next subsection). In that sense, these effective theories know a lot about the full theory. Of course, they are only effective field theories, but having several of them related by duality relations, make possible to define connections between the full theories to control their divergent behaviour (see section 6).

[2] In the paper by Cremmer, Julia & Scherk [2] only one sign has been used. I refer to [13] for a discussion of this sign freedom.

From now, I will only use the framework of the effective supergravity theories and their correspondences. These relations are the key point in the arguments to be developped in these notes. Superstring theory will be needed to derive these correspondances in the context of ultra-violet *finite* theories. Moreover superstring theories will allow us to write relations between supergravity theories with a really different spectrum. Despite, the superstrings will only appear dimly, it should be kept in mind that their very existence is needed.

- *The superalgebra and their central charges*

The super-Poincaré algebra in ten dimensions is generated by the translations P_μ ($\mu \in \{0,\ldots,9\}$), the Lorentz rotations $M_{\mu\nu}$ and by $\mathcal{N}_{10}$ real charges Q^I_α ($\alpha \in \{1,\ldots,32\}$, $I \in \{1,\ldots,\mathcal{N}_{10}\}$)[3] satisfying the the Majorana reality condition ${}^tQ^I C = \bar{Q}^I$ where C is the charge conjugation operator. $\Gamma_{11} = \Gamma^0 \cdots \Gamma^9$ ($(\Gamma_{11})^2 = 1$) is the chirality operator, the conditions $\Gamma_{11} Q^I_\pm = \pm Q^I_\pm$ reduce the number of component to sixteen. $\mathcal{P}_\pm = \frac{1}{2}(1 \pm \Gamma_{11})$ is the chirality projector.

The most general $\mathcal{N}_{10} = 1$ supersymmetric algebra is [8]

$$\begin{aligned}
[P_\mu, P_\nu] &= 0\\
[M_{\mu\nu}, P_\rho] &= P_\mu \eta_{\nu\rho} - P_\nu \eta_{\mu\rho}\\
[M_{\mu\nu}, M_{\rho\sigma}] &= M_{\nu\rho}\eta_{\mu\sigma} - M_{\mu\rho}\eta_{\nu\sigma} + M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\sigma}\eta_{\mu\rho}\\
[Q^+, P_\mu] &= 0\\
\{Q_+, Q_+\} &= (C\Gamma^\mu \mathcal{P}_+)\, P_\mu + (C\Gamma^{\mu\nu\rho\kappa\sigma}\mathcal{P}_+)\, Z^+_{\mu\nu\rho\kappa\sigma}\ .
\end{aligned} \tag{1.1}$$

Only the latest equation will of interest for us. The first term on the right hand side of the last equation is the usual generator of the translation, but the second deserves some explanations. This is needed by simple counting of the number of components, the left hand side has 136 independent component (recall that each Q^+ has sixteen real components) which are decomposed as ten components for the impulsion P_μ and 126 for the self-dual five-form $Z^+_{(5)}$. This term is not a central charge of the full algebra, being not a scalar under the Lorentz rotations, but is a central extension of the super-translation algebra generated by the first, penultimate and last equation of (1.1). The presence of the self-dual five-form $Z^+_{(5)}$ induces a contribution to the energy-impulsion tensor in the right-hand-side of Einstein equations. This is interpreted as the presence of an extended five dimensional object: a five-brane [14,15]. The reduced super-algebra, defined in the transverse space of

[3] $\mathcal{N}_{10}$ counts the number of real super-charges as calibrated in ten dimensions. This is related to the four dimensions counting by $\mathcal{N}_4 = 4\mathcal{N}_{10}$.

this five-brane, will take the form of a conventional supersymmetry algebra with a central charge given by the flux of the five-form on the world-volume of the five-brane:

$$Z^{+}_{(5)} = \mu_5 \int_{\Sigma_5} d\omega_5 \ .$$

In a quantum version of this extended object, the coupling to this five-form will appear as a Wess-Zumino term with the coefficient μ_5. Dirac quantization for extended objects relates this coefficient to the one of a one-brane. More generally, the presence of a p-brane induces a defect in space-time experiences by the phase of the wave function of a $6-p$-brane. Asking for the phase to change by a multiple of 2π after a close trajet around the p-brane leads to the quantization condition $2\kappa_{10}^2\mu_{(p)}\mu_{(6-p)} = 2\pi n$, $n \in \mathbb{Z}$ [16]. The appearance of other form in super-algebra has to be thought as generalization of this example.

The maximal extension (with respect to the sub-algebra of super-translations) of the non-chiral type IIa super-Poincaré $\mathcal{N}_{10} = 2$ algebra generated by the real super-charges $Q \stackrel{\text{def}}{=} (Q^+_\alpha, Q^-_\alpha)$ ($\alpha = 1, \dots, 16$) of opposite chiralities is

$$\begin{aligned}\{Q, Q\} = (C\Gamma^\mu)P_\mu + (C\Gamma_{11})Z + (C\Gamma_{11}\Gamma^\mu)Z_\mu + (C\Gamma^{\mu\nu})Z_{\mu\nu} \\ + (C\Gamma_{11}\Gamma^{\mu\nu\rho\sigma})Z_{\mu\nu\rho\sigma} + (C\Gamma^{\mu\nu\rho\kappa\sigma})Z_{\mu\nu\rho\kappa\sigma}.\end{aligned} \tag{1.2}$$

The left hand side of this equation is a symmetric matrix of rank 32 with real entries, so has 528 independent components. These are decomposed as $528 = 10 + 1 + 10 + 45 + 210 + 252$: the impulsion P_μ, a charge $Z_{(0)}$ carried by the Dzero-brane (point-like object), $Z_{(1)}$ carried by a string, $Z_{(2)}$ carried by a Dtwo-brane, $Z_{(4)}$ by a Dfour-brane and $Z_{(5)}$ by a (NS) five-brane.

The chiral version of the previous equation for the $\mathcal{N}_{10} = 2$ chiral algebra of type IIb generated by two Majorana-Weyl spinors Q^i_α ($\alpha = 1, \dots, 16$) of same chirality $\mathcal{P}_- Q^i = 0$ ($i = 1, 2$) is

$$\begin{aligned}\{Q^i, Q^j\} = \delta^{ij}(C\Gamma^\mu\mathcal{P}_+)P_\mu + (C\Gamma^\mu\mathcal{P}_+)Z^{ij}_\mu + \varepsilon^{ij}(C\Gamma_{11}\Gamma^{\mu\nu\rho}\mathcal{P}_+)Z_{\mu\nu\rho} \\ + \delta^{ij}(C\Gamma^{\mu\nu\rho\kappa\sigma}\mathcal{P}_+)Z^+_{\mu\nu\rho\kappa\sigma} + (C\Gamma^{\mu\nu\rho\kappa\sigma}\mathcal{P}^+)Z^{ij}_{\mu\nu\rho\kappa\sigma}.\end{aligned} \tag{1.3}$$

The left hand side of this equation has $2 \times \frac{16\times 17}{2} + 16^2 = 528$ independent components decomposed as ten for the impulsion P_μ, 2×10 for symmetric traceless $SO(2)$ tensor $Z^{ij}_{(1)}$, 120 for $Z_{(3)}$, and three self-dual five-forms with 126 degrees of freedom each: $Z^+_{(5)}$ and $Z^{ij}_{(5)}$ ($i, j = 1, 2$). The charge $Z^{ij}_{(1)}$ is carried by a doublet made of a string and a Done-brane, $Z_{(3)}$ is carried by a Dthree-brane, $Z^+_{(5)}$ and $Z^{ij}_{(5)}$ are carried by the NS and Dfive-branes.

Finally, the $\mathcal{N}_{11} = 1$ super-algebra in eleven dimensions generated by a Majorana spinor Q_α ($\alpha = 1, \dots, 32$) given by

$$\{Q, Q\} = (C\Gamma^M)P_M + (C\Gamma^{MN})Z_{MN} + (C\Gamma^{MNPQR})Z_{MNPQR}, \tag{1.4}$$

with $M, N, \dots$ run over $\{1, \dots, 11\}$. Again, the left-hand side has 528 real independent components decomposed as $528 = 11 + 55 + 462$: with the charge $Z_{(2)}$ carried by the Mtwo-brane and a charge $Z_{(5)}$ by the Mfive-brane.

• Solutions of p-branes

When considering the classical solution of an extended $p+1$ dimensionnal plane in spacetime, we can start with an ansatz for which the metric respects the geometry of the configuration [15]

$$ds^2 = e^{2A} dx^2 + e^{2B} dy^2$$
$$C^{(p+1)}_{\mu_1\cdots\mu_{p+1}} = e^C \, \varepsilon_{\mu_1\cdots\mu_{p+1}}$$

where x and y are respectively the coordinates parallel and transverse to the p-brane, there is constant charge under a $p+1$-form $C_{(p+1)}$ carried by this (electric) object, and all the other fields set to zero. In particular all the fermions are zero. This ansatz contains the minimal requierements to fix the geometry of the problem. The $SO(1,9)$ symmetry of the empty space is broken to $SO(1,p)\times SO(9-p)$ by the ansatz.

Solving the equations of motion for the bosonic fields, shows that A, B and C depend on the dilaton, with the constrainted that the fonction $\exp(-\phi(y))$ is harmonic in the transverse distance $y=\sqrt{y^m y_m}$. The theory having supersymmetry there are constraints on the existence of Killing spinors compatible with this background. The equation is deduced by asking that the supersymmetry variation of the gravitino ψ_μ is zero [17]

$$\begin{aligned} \delta_\epsilon\psi_\mu &= \partial_\mu\epsilon - \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon + \frac{(-1)^p}{8(p+2)!}e^\phi F_{\mu_1\cdots\mu_{p+2}}\Gamma^{\mu_1\cdots\mu_{p+2}}\Gamma_\mu\epsilon'_{(p)} \\ \delta_\epsilon\lambda &= \Gamma^\mu\partial_\mu\phi\epsilon + \frac{3-p}{4(p+2)!}e^\phi F_{\mu_1\cdots\mu_{p+2}}\Gamma^{\mu_1\cdots\mu_{p+2}}\epsilon'_{(p)} \ . \end{aligned} \tag{1.5}$$

D_μ is the covariant derivative, $F_{(p+2)} = dC_{(p+2)}$ is the field strength of the $C_{(p+1)}$ under which is charged the extended object and $\Gamma^{(p+1)}$ is the anti-symmetric product (with weight one) of gamma matrices, ω is the spin connection and

$$\epsilon'_{(0,4,8)} = \epsilon, \quad \epsilon'_{(2,6)} = \Gamma_{11}\epsilon, \quad \epsilon'_{(-1,3,7)} = i\epsilon, \quad \epsilon'_{(1,5)} = i\epsilon^*$$

Depending on the chirality of the theory and the number of supersymmetries, there are two kinds of solutions: (1) the *Neveu-Schwarz* solutions valid only for objects of dimensions $p=1$ and $p=5$ (the string and the five-brane), and (2) *Dirichlet* solutions valid for every p compatible with the chirality of the theory ($p=0,2,4,6,8$ for (1.2) and $p=-1,1,3,5,7$ for (1.3)).

For $\mathcal{N}_{10}=2$ supersymmetrical theories, the parameter splits naturally into a left ($\varepsilon_{\rm L}$) and right ($\varepsilon_{\rm R}$) moving parameter : $\delta_{susy} = \varepsilon_{\rm L}Q_{\rm L} + \varepsilon_{\rm R}Q_{\rm R}$. Neveu-Schwarz solution to (1.5) can be solved if the left and right moving Killing spinors satisfy

$$\mathcal{P}_{p+1}\varepsilon_{\rm L} = 0, \qquad \mathcal{P}_{p+1}\varepsilon_{\rm R} = 0 \ , \tag{1.6}$$

where $\mathcal{P}_{p+1}$ is a projector.

Dirichlet solutions to (1.5) are given by an equation relating the left and right moving Killing spinors

$$\Gamma^{\mu_1\cdots\mu_{p+1}}\varepsilon_{\mathrm{L}} = \varepsilon_{\mathrm{R}} \ , \tag{1.7}$$

$\Gamma^{\mu_1\cdots\mu_{p+1}}$ is the antisymmetric product of $p+1$ gamma matrices with the indices μ_i restricted to the values $0,\cdots,p$ of the along the word-volume of the $p+1$-brane.

The remarkable difference between (1.6) and (1.7) is that the later breaks the super-Poincaré algebra $SO(1,9)\to SO(1,p)\times SO(9-p)$ with preserving a sub $\mathcal{N}_{10}=1$ algebra generated by $\delta_{suzy} = \varepsilon_{\mathrm{L}}\,(Q_{\mathrm{L}} + \Gamma^{\mu_1\cdots\mu_{p+1}}Q_{\mathrm{R}})$, associated with the dynamics of the small open string with end points attached to the D-brane [18].

• Supertraces over helicities

A generic extended $\mathcal{N}_4$ supersymmetry algebra with central charges takes the form [19]

$$\begin{aligned}\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta}Z^{IJ}, \quad \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = \epsilon_{\alpha\beta}\bar{Z}^{IJ}\\ \{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = \delta^{IJ}2\sigma^\mu_{\alpha\dot\beta}P_\mu.\end{aligned} \tag{1.8}$$

The indices $\alpha,\beta=1,2$ belong to the complex representation of $\mathrm{Sl}(2,\mathbb{C})$ and $\dot\alpha,\dot\beta=1,2$ to the inequivalent complex conjugate representation. $\sigma^\mu_{\alpha\dot\beta}$ is the usual Clebsch-Gordan coefficient for the tensor product decomposition of $Q\otimes\bar{Q}$ on the vectorial representation of P_μ SO(1,3).

First I consider the case without central charges [19,20]. The massive representation are given with respect to the rest frame for the particle $P=(-M,0)$,

$$\begin{aligned}\{Q^I_\alpha, Q^J_\beta\} = \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = 0\\ \{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2\delta^{IJ}\delta_{\alpha\dot\beta}M.\end{aligned}$$

I define the normalized creation and annihilation operators

$$a^I_\alpha \stackrel{\text{def}}{=} \frac{1}{\sqrt{2M}}Q^I_\alpha, \qquad a^{\dagger\, I}_\alpha \stackrel{\text{def}}{=} \frac{1}{\sqrt{2M}}\bar{Q}^I_{\dot\alpha}.$$

Acting on the ground state $|\Omega\rangle$ annihilated by the a^I_α, one builds a Fock space of dimension $2^{2\mathcal{N}_4}$. If the ground state has a spin j, acting on it with n creation operators and m annihilation operators, one obtains a state with spin $j+n-m$.

For massless particles, the rest frame is given by $P=(-E,0,0,E)$. The generators of the super-algebra are given by

$$\{Q^I_\alpha, Q^J_\beta\} = \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = 0$$
$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2\delta^{IJ}\begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}.$$

I defining for this case the creating and annihilating operators

$$a^I \stackrel{\text{def}}{=} \frac{1}{\sqrt{2E}} Q^I_1, \qquad a^{\dagger\, I} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2E}} \bar{Q}^I_1 \ ,$$

the Q_2 charges have a trivial action. Acting on the ground state $|\Omega\rangle$ annihilated by the a^I, one builds a Fock space of total dimension $2^{\mathcal{N}_4}$. The spin content of a massless representation build from a ground-state with helicity λ is

$$M_0^\lambda : [\lambda \pm 2] + 8\,[\lambda \pm \tfrac{3}{2}] + 28[\lambda \pm 1] + 56\,[\lambda \pm \tfrac{1}{2}] + 70\,[\lambda] \ .$$

I consider now the case with central charges Z and $\bar{Z}$. Rotating the indices I, J with an element $U(\mathcal{N}_4)$ it is always possible to write the central charges in the canonical form ($a = 1, 2$ et $m = 1, \ldots, \lfloor \mathcal{N}_4/2 \rfloor$) [21]

$$Z^{a\,m} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathrm{Diag}(z_1, \ldots, z_{\frac{\mathcal{N}_4}{2}}) \qquad \text{for even } \mathcal{N}_4$$
$$Z^{a\,m} = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathrm{Diag}(z_1, \ldots, z_{\lfloor \frac{\mathcal{N}_4}{2} \rfloor}) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for odd } \mathcal{N}_4 \ .$$

The anti-commutation relations become

$$\{Q^{a\,m}_\alpha, Q^{b\,n}_\beta\} = z_m \epsilon_{\alpha\beta} \epsilon^{ab} \delta^{mn}$$
$$\{Q^{a\,m}_\alpha, \bar{Q}^{b\,n}_{\dot\beta}\} = 2M \delta^{mn} \delta^{ab} \delta_{\alpha\beta}.$$

Defining creating and annihilating operators by $a^m_\alpha \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\left[Q^{1m}_\alpha + \epsilon_{\alpha\beta} Q^{2m}_\beta\right]$ et $b^m_\alpha \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\left[Q^{1m}_\alpha - \epsilon_{\alpha\beta} Q^{2m}_\beta\right]$, we get

$$\{a^m_\alpha, a^n_\beta\} = \{a^m_\alpha, b^n_\beta\} = \{b^m_\alpha, b^n_\beta\} = 0$$
$$\{a^m_\alpha, a^{\dagger n}_\beta\} = \delta_{\alpha\beta}\delta^{mn}\,(2M + z_n), \quad \{b^m_\alpha, b^{\dagger n}_\beta\} = \delta_{\alpha\beta}\delta^{mn}\,(2M - z_n)\,.$$

Positivity of the energy imply the positivity for the right hand side of the previous equations, and so $2M \geq \mathrm{Max}(z_n)$. This is the so-called Bogomol'ny-Prasad-Sommerfeld inequality. When M is equal to some central charges (say p of them) the operators b^m et $b^{\dagger\, m}$ with $m = 1, \ldots, p$ decouple from the rest of the algebra. The Fock space has now the reduced dimension $2^{2(\mathcal{N}_4 - p)}$. For $\mathcal{N}_4 = 8$ theories we can have up to four central charges $z_1 \geq z_2 \geq z_3 \geq z_4$. I introduce some terminology for the various kind of representations of these algebra.

First, the celebrated massive *short multiplets* or *half-BPS* multiplets obtained when $2M = z_{1,2,3,4}$. Their associated representation has the following spin content

$$S_j : [j] \otimes \left([2] + 8\,[\tfrac{3}{2}] + 27\,[1] + 48\,[\tfrac{1}{2}] + 42\,[0]\right) \ ,$$

with dimension $(2j+1)\times 2^8$ if built on a ground-state with a spin j. They have the same dimension as the massless representation and can be decomposed as

$$S_j = \sum_{\lambda=0}^{j} M_0^\lambda \ . \tag{1.9}$$

This relation between the massive short multiplets and the massless representations is at the core of the reinterpretation of D-branes states[4] as super-gravitons excitations in eleven dimensions [9]. I will heavily use this connection in section 6 when discussing unitarity technics in $\mathcal{N}_4 = 8$ supergravity theories.

If M is bigger than all the z_i, we have the *long multiplets*. They break all the supersymmetries and contain $(2j+1)\times 2^{16}$ states.

If $2M = z_1 > z_{2,3,4}$, or $2M = z_{1,2} > z_{3,4}$ or $2M = z_{1,2,3} > z_4$ we have intermediate multiplets. They break respectively seven, six and five supersymmetries and contain $(2j+1)\times 2^{14}$, $(2j+1)\times 2^{12}$ and $(2j+1)\times 2^{10}$ states.

Superstring with their left-moving states on their ground-states and their right-moving states in a winding states with $P_L^2 - p_R^2 = 4N > 0$ belong to the intermediate multiplets with size $(2j+1)\times 2^{12}$. This configuration (appearing when at least one dimension is compactified, *i.e.* $d \leq 9$) preserves 1/4th of the supersymmetries.

Quantum loop corrections to the effective action of supersymmetric theories can be expressed as super-traces over helicities in the context of the gauge theory [22,23,24] or in supergravity theory [25].

In super Yang–Mills theory they take the form

$$\mathcal{L}_{\text{eff}} = \int_0^\infty \frac{dt}{t}\, \mathrm{Str}\left((tQ^aF_a)^4\, e^{-tS_{\text{SYM}}}\right)$$

where t is the Schwinger time, and F_a and Q^a are in the adjoint representation of the gauge group. The super-trace operator, Str, is defined as: $\mathrm{Str}(f) = \mathrm{tr}_{\text{boson}}(f) - \mathrm{tr}_{\text{fermion}}(f)$. Each trace are evaluated in the adjoint representation with summation over all the states running in the loop. The equivalent expression for supergravity theories will be introduced in the next section. Define the super-trace over the helicity λ for a Fock space with a ground-state with spin $[j]$

$$B_{2n}([j]) \stackrel{\text{def}}{=} \mathrm{Str}_{[j]}\left(\lambda^{2n}\right) = \mathrm{tr}_{[j]}\left((-)^{2\lambda}\,\lambda^{2n}\right) . \tag{1.10}$$

The matching between bosonic and fermionic degeneracies imply that $B_0([j]) \equiv 0$ for any spin j. For $\mathcal{N}_4 = 8$ theories, we have $B_n([j]) = 0$ for $n = 0, \ldots, 6$ for all j. For $n \geq 8$, B_n is no longer zero for the massless and the short multiplets. We still have $B_{10}(I_1^j) = 0$ but non-zero for $n \geq 10$ for the first intermediate multiplet. $B_{8,10}(I_2^j) = 0$ and non-zero for higher n for the second intermediate multiplet. $B_{8,10,12}(I_3^j) = 0$ for non-zero for higher n for the third intermediate multiplet. And $B_{8,10,12,14} = 0$ for the long multiplet and non-zero for n bigger than fourteen [20].

[4] They are half-BPS states obtained for non-zero values of any $Z_{(n)}$ in (1.2) or (1.3).

2. The bound state problem

The presence of a membrane (Mtwo-brane) in eleven dimensions could have cured the ultra-violet divergent behaviour of the supergravity in the same way superstrings cure the ten dimensional supergravity theory. Unfortunately the light-cone quantization of the membrane is a $SU(N)$ quantum mechanics with a continuous spectrum without mass-gap of non-normalizable states. This killed all hopes in getting any fundamental theory from the membrane [26]. The strong coupling regime of superstring theories is govern by the problem of bound states of D-branes, that are in this regime, bound states at threshold of super-gravitons in eleven dimensions. The very existence of a M-theory has its root in the existence, inside the annoying continuous spectrum of the membrane, of these discrete bound states [27,28,29]. The dynamics of the bound states are given by a Matrix model obtained by dimensional reduction from the super Yang–Mills theory in ten dimensions [27]. It is believed that this matrix model is a fundamental building block of the non-perturbative superstring physics [30,31]. I will now present some aspects of it in its zero-dimensional version.

A D-instanton (or D(-1)-brane) corresponds to the extreme case where the dimensional reduction is complete and the world-volume is a single point in Euclidean spacetime. Correspondingly, rather than finite mass or tension, a single D-instanton has finite action $S_{\rm cl} = 2\pi(e^{-\phi} - ic^{(0)})$ where $c^{(0)}$ is the Ramond-Ramond scalar of the IIb theory. The collective coordinates of a charge-k D-instanton correspond to a $U(k)$ vector multiplet of the ten dimensional super Yang–Mills theory. The diagonal components of the ten-dimensional gauge field A_μ specify the location of k D-instantons in $\mathbb{R}^{10}$. As we dimensionally reduce to $d = 0$, these degrees of freedom are numbers rather than fields. In addition to a constant part equal to kS_{cl}, the action of a charge-k D-instanton also depends on the collective coordinates via the following action

$$S_k = -\frac{1}{2g_0^2}\mathrm{tr}_k\,[A_\mu, A_\nu]^2 + \frac{1}{g_0^2}\mathrm{tr}_k\,\bar\Psi\Gamma_\mu[A_\mu,\Psi], \tag{2.1}$$

where $g_0 \sim (\alpha')^{-2}$.

In addition to the manifest $SO(10)$ symmetry under ten-dimensional rotations, the action is trivially invariant under translations of the form $A_\mu \to A_\mu + x_\mu\mathbb{I}$. Hence $k^{-1}\mathrm{tr}\,A_\mu$, which corresponds to the Abelian factor of the $U(k)$ gauge group, is identified with the centre-of-mass coordinate of the charge k D-instanton. The fermionic symmetries of the collective coordinate action are,

$$\begin{aligned} \delta A_\mu &= i\bar\eta\Gamma_\mu\Psi\ ,\\ \delta\Psi\ &= -[A_\mu, A_\nu]\Gamma^{\mu\nu}\eta + \mathbb{I}\,\varepsilon\ , \end{aligned} \tag{2.2}$$

where $\Gamma_{\mu\nu} = i[\Gamma_\mu,\Gamma_\nu]/4$. The sixteen components of the Majorana-Weyl $SO(10)$ spinor ε correspond to the sixteen zero modes of the D-instanton configuration generated by the action of the $D = 10$ supercharges. Like the bosonic translation modes, these modes live in the Abelian factor of the corresponding $U(k)$ field, $k^{-1}\mathrm{tr}\,\Psi$. In contrast, the Majorana-Weyl spinor η parametrises the sixteen supersymmetries left unbroken by the D-instanton.

In ordinary field theory, instantons yield non-perturbative corrections to correlation functions via their saddle-point contribution to the Euclidean path integral. In the semi-classical limit, the path-integral in each topological sector reduces to an ordinary integral over the instanton moduli space. The extent to which similar ideas apply to D-instantons is less clear, in part because string theory lacks a second-quantised formalism analogous to the path integral. Despite this, there is considerable evidence that D-instanton contributions to string-theory amplitudes also reduce to integrals over collective coordinates at weak string coupling [32]. In this case the relevant collective coordinates are the components the ten-dimensional $U(k)$ gauge field A_μ and their super-partners Ψ. According to Green and Gutperle [32], the charge-k D-instanton contributions to the low-energy correlators of the IIb theory are governed by the partition function,

$$\mathcal{Z}_k = \frac{1}{\mathrm{Vol}\,U(k)} \int_{U(k)} d^{10}A\, d^{16}\Psi\ \exp -S_k\ . \tag{2.3}$$

This partition function can be thought of as the collective coordinate integration measure for k D-instantons. In particular, the leading semi-classical contribution of k D-instantons to the correlators of the low-energy supergravity fields is obtained by inserting the classical value of each field in the integrand of (2.3). This interpretation has been confirm by showing that $\mathcal{Z}_k$ reproduce the measure factor $\mu_2(k) = \sum_{d|k} d^{-2}$ of D-instantons effects in supergravity. Because of the symmetries described above, the collective coordinate action S_k does not depend on the $U(1)$ components of the fields A_μ and Ψ. Hence, to obtain a non-zero answer, the inserted fields must include at least sixteen fermions to saturate the corresponding Grassmann integrations. As in field theory instanton calculations, the resulting amplitudes can be interpreted in terms of an instanton-induced vertex in the low-energy effective action. The spacetime position of the D-instanton, x_μ, is interpreted as the location of the vertex. The $t_8t_8R^4$ terms in the IIb effective action and its supersymmetric completion, satisfy this property and will be discussed in section 7. It is interesting to understand along the same line of reasoning the supergravity measure factor for 1/4th-BPS states $\mu_4(k) = \sum_{d|k} d^{-4}$ contributing to the higher-order derivative $D^4t_8t_8R^4$ obtained in [33].

The integral over the $SU(k)$ part of (2.3) is unexpectedly convergent. The action (2.1) has flat directions parametrised by the eigenvalues of mutually commuting matrices. These flat directions, representing valleys of quadratic potential, are not lifted by quantum correction thanks to the zero-point energy cancellation between the bosonic and fermionic variables, and are responsible for the continuous spectrum of the model [26]. These valleys are permuted by the elements of the Weyl group of $SU(k)$. The gauge symmetry and the supersymmetry are enough to cure the integral from ultra-violet divergences from the flat directions, and the integral is finite [28,32,34].

It is interesting to remark that the supersymmetry transformations (2.2) involve two parameters ϵ and η to which we can assign a $U(1)$ R-charge [35]. Such that the fields of the model will acquire a $U(1)$ weight. In fact, only the fluctuations of the field have a proper weight, the classical value does not need to have a defined weight. This symmetry is part of the classical symmetry of the chiral supergravity in ten dimensions $G = Sl(2,\mathbb{R})/SO(2)$. The model (2.1) does not seem to have any invariance over $Sl(2)$ group. But we will see in section 7 that this model will appear as a saddle-point for the gauge configurations connected to the geometry of the D-instanton. If the matrix model contains some deep informations about the non-perturbative supertring physics, it should possess this $Sl(2,\mathbb{Z})$ symmetry.

3. Unitary and supergravity theories

Gravity like every theory with gauge invariance needs a rather involved treatment when considered front the point of view of Feynman diagrams. For a theory presenting some gauge invariance, it is necessary to separate the physical and non-physical degrees of freedom. In a very naive way of quantising such a model, one picks a peculiar configuration of the fields constrained by a gauge choice, and applies the usual Feynman rules to compute S-matrix elements. At the end, the result should be gauge invariant, and independent of the configurations chosen for the fields. The gauge independence results from various non-trivial cancellations between diagrams. In particular, inside loops some unphysical contributions encoded in determinant have to cancel against non-physical parts in propagators. This is conveniently tackled by introducing Fadeev-Popov ghosts which run in the loops and cancel the unphysical contributions. At the end is left some simple integrals, $I_\alpha(s,t,u)$, multiplied by some coefficients, C_α containing the polarisations and the momenta of the external states:[5]

$$\mathcal{A}_n^{\ell-\text{loop}}(1,\cdots,n) = g^{n-\ell} \sum_{\alpha\in\mathcal{F}} C_\alpha(1,\cdots,n) I_\alpha(s,t,u) \ , \tag{3.1}$$

$\mathcal{F}$ is a finite family of indices and g is a coupling constant. In the usual Feynman diagram approach this reduction appears only at the end of tedious computations. If the situation is

[5] A particular case of this general decomposition is the so-called colour decomposition in Yang–Mills gauge theory. The scattering amplitudes between charged species in open strings are naturally composed according the ordering of the Chan–Paton factors, which is the colour decomposition of the amplitude and the integrals I_α are interpreted as scalar loop integrals. My interest being only for gravitational theories there are no colour quantum numbers, and the results will be decomposed according traces over helicities. Nevertheless, I will use the words 'colour-like decomposition'.

complicated for Yang–Mills gauge theory, it is even worse for gravitation having not a finite number of vertices in the theory. And in the case we are interested in, the supergravity theories, additional combinations of super-partners have to be considered. Again a lot of cancellations between diagrams where bosonic and fermionic fields run in the loops are expected. A direct computation of all these diagrams gets rapidly complicated and involved a huge number of terms. We will see that for the case of amplitudes, where a number of fermions less than the total numbers of supersymmetries of the theory are involved, the coefficients C_α reduce to a small set with fixed tensorial structure. That will be, in particular, the case of the one- and two-loop amplitude discussed in section 6.

Unitary relations are expressed on the S-matrix elements and are just a transcription of the unitarity of the scattering matrix ($S = \mathbb{I} + iT$):

$$S^\dagger = S \iff 2\Im\mathrm{m} T_{ij} = \sum_\ell T^\dagger_{\ell j}\, T_{\ell i} \ . \tag{3.2}$$

These relations are by definition gauge invariant and respect the decomposition (3.1) of the amplitudes and are independent of any perturbation scheme. Plugging (3.1) in (3.2), the computation reduces to fusion relations between the coefficients

$$C_\alpha(\ell, j)^\dagger\, C_\beta(\ell, i) = C_\gamma(i, j) F_\gamma(\hat{s}, \hat{t}, \hat{u}), \tag{3.3}$$

where $F_\gamma(\hat{s}, \hat{t}, \hat{u})$ are functions of the Mandelstam variables constructed with the momenta k_i and ℓ, only. Again for special classes of amplitudes, to be discussed in section 6, the elementary integrals $I_\alpha(s, t, u)$ will have the interpretation of loop integrals for scalar-φ^3 field theory. If the colour-like coefficients satisfy simple enough fusion rules, then loop amplitudes can be constructed recursively using unitary methods on the scalar loop integrals. Such special cases are called cut constructible amplitudes, following a denomination of Bern, Dixon & Kosower [36]. The arguments to be presented apply to the cases of amplitudes with massless external particles, and half-BPS states thanks to the relation (1.9).

These authors and their collaborators are mainly interested in gauge theory, three jets physics, for example. My interest is mostly on the structure of supergravity with a special interest in understanding some properties of the elusive M-theory. I will set myself in the framework of supergravities theories with thirty-two real supercharges, formulated as the eleven dimensions supergravity, or as the low-energy limit of the close type II superstring theories (cf. section 1). Despite the relation (3.2) is independent of any perturbative expansion there is no clear formulation, up to my knowledge, of unitary relation for contributions involving non-perturbative states. Fortunately, the non-perturbative string states to be considered are just ordinary excitations of the metric for the eleven dimension theory. The possible states associated with wrapped Mtwo- or Mfive-brane will not be considered.

Applying this relation recursively one-loop and two-loop diagrams for the (effective) eleven-dimension supergravity theory will be constructed. Compactifying one or two dimensions, these results will be reinterpreted in the context of the ten or nine dimensional effective supergravity theories of type IIa or IIb. For the particular case of four-graviton scattering the result is the same up-to and including two loops, irrespectively of the chirality difference between these theories. This identity will make possible to fix the value of the counter-terms for the cut-off regularization scheme.[6] This equivalence is the key argument in these considerations, if we had only one effective theory we would have not been able to fix the value of the counter-terms.

4. Superstring theory considerations

I will first review the results for the four gravitons scattering in type II superstring theories. String theory formulated in the covariant or the light-cone gauge scheme, gives a result directly expressed in the form (3.1). This simple reduction of the amplitude results from the interplay between the gauge invariance and the supersymmetries of the theory.

For what concerns the supersymmetries, superstrings are extended objects that break sixteen real (half of the) supercharges, leaving sixteen supersymmetries linearly realized (the fermionic zero modes). The vertex operators for the graviton, in the zero-ghost picture, are

$$V_g(z,\bar z) = \zeta_{\mu\nu} \,{}^\times_\times\, \left(\partial x^\mu(z) + ik\cdot\psi\psi^\mu\right)\left(\bar\partial x^\nu(\bar z) + ik\cdot\psi\psi^\nu\right)\, e^{ik\cdot X} \,{}^\times_\times \tag{4.1}$$

having four of them, the sixteen fermionic zero modes of the vacuum are soaked up by the sixteen fermions ψ from the vertex operators, leaving a correlation of the plane wave factors

$$\left\langle \prod_{\alpha=1}^{4} V_g^{(\alpha)}(z_\alpha,\bar z_\alpha)\right\rangle_{\ell-\mathrm{loop}} \sim \left(\widehat{K}^2 + \mathrm{odd}-\mathrm{spin}\right)\times \left\langle \prod_{i=1}^{4} \,{}^\times_\times\, e^{ik^\alpha\cdot X(\alpha)} \,{}^\times_\times \right\rangle_{\ell-\mathrm{loop}} , \tag{4.2}$$

times the factor $\widehat{K}^2 = t_8 t_8 R^4$ which is the kinematic factor

$$\widehat{K}^2 = t^{\mu_1\cdots\mu_8} t^{\nu_1\cdots\nu_8} \prod_{r=1}^{4} \zeta^{(r)}_{\mu_r\nu_r} k^{(r)}_{\mu_{r+4}} k^{(r)}_{\nu_{r+4}} \ . \tag{4.3}$$

The correlation function multiplying this gauge factor does not depend anymore on the spin of the incoming momenta, and is a function of the Mandelstam invariants s, t and u only. The gauge invariance of the theory is encoded in the $t_8 t_8 R^4$ prefactor which will be the only color-like coefficient C_α for these amplitudes. In particular all the supersymmetric Ward identities, obtained by commuting supercharges inside the correlation functions, will

[6] see section 4 on this equivalence between type IIa and IIb theories.

be expressed as bootstrap identities (3.3) on $t_8 t_8 R^4$ factors. This kinematic factor is closely related to the super-trace over helicity $B_8([2])$ for $\mathcal{N}_4 = 8$ theory defined in (1.10),

$$\widehat{K}^2 = \mathrm{Str}_{[2]} \left(\prod_{i=1}^{4} \frac{1}{4} S\gamma^{a_i b_i} S \times \frac{1}{4} \tilde{S}\gamma^{a'_i b'_i} \tilde{S} \right) \prod_{r=1}^{4} \zeta^{(r)}_{a_r a'_r} k^{(r)}_{b_r} k^{(r)}_{b'_r}$$

The remaining part of the amplitude contains the analytic properties of the loop amplitudes.

This set-up will be duplicated when considering a first quantized version of the one-loop amplitudes in eleven dimensions.

- Tree-level and one-loop amplitude

At tree-level there is no difference between the four graviton scattering amplitude computed in the non-chiral type IIa or in the chiral type IIb theory. The difference between these two theories is a choice of sign for the GSO projection for the odd-spin structure contributions. Since we are only considering particles in the Neveu-Schwarz sector the differences can only arise from the odd-spin structures sector. At one-loop, in ten or nine dimensions the amplitudes are the same, in eight dimensions they only differ by a total derivative.[7] For trivial backgrounds this difference will be ignored (see section 2.3 of [33]). Up to two loops there is no contributions from the odd-spin structures terms to the four gravitons scattering amplitude, because of the conservation of the momenta.[8] After, three loops the amplitude receive contributions from the odd-spin structure, I will comment on that at the end of next section.

The sum of the tree-level and one-loop contributions to the four-graviton amplitude in ten dimensions has the form [6,39,40],

$$\mathcal{A}^{\mathrm{string}}_4 = \kappa^4_{10} \widehat{K}^2 \left[-\frac{1}{\hat{g}^2} T(s,t,u) + \frac{1}{2^5 \pi^6} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F(\tau,\bar{\tau};s,t,u) \right], \tag{4.4}$$

where the functions T and F contain the dependence on the Mandelstam invariants of the tree-level and one-loop terms, respectively, $2\kappa^2_{10} = (2\pi)^7 l_s^8$ is defined as in [41], $\hat{g} = e^{\phi}\kappa_{10}/l_s^8$

[7] The resulting difference is an antisymmetric combination of the Riemann tensor $Z = R^{ab}_{[ab} R^{cd}_{cd} R^{ef}_{ef} R^{gh}_{gh]}$ which is the bosonic part of supersymmetric invariant up to a total derivative. The Riemann tensor being the covariant derivative of the spin connection $R^{ab}_{\mu\nu} = D_{[\mu}(\omega)\omega^{ab}_{\nu]}$, Z is invariant for any variation of ω thanks to the Bianchi identity $D_{[\mu} R^{ab}_{\nu\rho]} = 0$.

[8] This symmetry between the type IIa and IIb theories for the four gravitons at one-loop and two loops is related to the T-duality correspondance between the two theories. It should be kept in mind that the states involved in the amplitude in both side belong to different multiplets of the relevant supergravity theories. So this symmetry is really non-trival (and not true for more than 4 gravitons amplitudes) and no self-dual radius, where the two theories coulb be identified, exists [37,38].

is the loop expansion parameter and $d^2\tau = d\tau_1 d\tau_2$. The relative coefficient between the tree-level and one-loop amplitude has been fixed with unitarity arguments [42].

The function T contains the dynamical part of the tree amplitude for the elastic scattering of two gravitons in either type II theory and is given

$$T = \frac{64}{l_s^6 stu} \frac{\Gamma(1-\frac{l_s^2}{4}s)\Gamma(1-\frac{l_s^2}{4}t)\Gamma(1-\frac{l_s^2}{4}u)}{\Gamma(1+\frac{l_s^2}{4}s)\Gamma(1+\frac{l_s^2}{4}t)\Gamma(1+\frac{l_s^2}{4}u)} \ . \tag{4.5}$$

Thus, the low energy expansion of the amplitude begins with the terms,

$$T = \frac{64}{l_s^6 stu} + 2\zeta(3) + \frac{\zeta(5)}{16} l_s^4 (s^2 + t^2 + u^2) + \cdots \tag{4.6}$$

The dynamical factor F in the loop amplitude is given in terms of the scalar Green function on the torus $\ln \chi_{ij}$,

$$F(\tau, \bar{\tau}; s, t, u) = \int_{\mathcal{T}^2} \frac{d^2\tau}{\tau_2^2} \prod_{i=1}^{3} \frac{d^2\nu^{(i)}}{\tau_2} \left(\chi_{12}\chi_{34}\right)^{l_s^2 s} \left(\chi_{14}\chi_{23}\right)^{l_s^2 t} \left(\chi_{13}\chi_{24}\right)^{l_s^2 u} \ . \tag{4.7}$$

The low-energy expansion of this expression is considered in [43] where the first few terms are shown to be,

$$\mathcal{A}_4^{1-\text{loop}} = \widehat{K}^2 \left(\frac{\pi}{3} + \frac{l_s^2}{16} I_{nonan\,1} + 0 \times l_s^4 (s^2 + t^2 + u^2) + \cdots \right) . \tag{4.8}$$

$I_{nonan\,1}(s,0,0) \sim s\ln(s)$ is the first normal threshold term encountered. The field theory limit of these expressions are obtained by neglecting the size of the string, $l_s \to 0$, [9]

$$\mathcal{M}^{\text{tree}}(1,2,3,4) \sim -\kappa_{10}^2 e^{-2\phi} \frac{\widehat{K}^2}{l_s^6 stu} \tag{4.9}$$

for the tree-level term, and

$$\mathcal{M}^{1-\text{loop}}(1,2,3,4) \sim -i\kappa_{10}^4 \times \widehat{K}^2 \times \Big(I^{(1)}(s,t) + I^{(1)}(s,u) + I^{(1)}(t,u) \Big) \tag{4.10}$$

for the one-loop contribution. It has been first realized by Green, Schwarz & Brink [45] that the field theory limit of the string loop amplitude involves the simple one-loop scalar-φ^3 integral:

$$I^{(1)}(s,t) = \int d^{10}p \, \frac{1}{p^2(p+k_1)^2(p+k_1+k_2)^2(p-k_4)^2} \tag{4.11}$$

This is how supergravity reduces the amplitude into the form (3.1).

[9] I will be rather sloppy with the normalizations; but they are needed for proper unitarity considerations. For accurate numbers consult [44,33].

5. Non-perturbative contributions

The non-perturbative superstring states are described by the D-branes. A D-brane breaks half of the supersymmetries, leaving sixteen fermionic zero modes. The remaining half belongs to short multiplets of the supersymmetry algebra. Short multiplets contains 256 states related by transformations generated by the sixteen broken supersymmetries. Any short multiplet can be decomposed on the massless representations of the super-algebra. So short multiplets comprise massless string states and D-branes states.

Since short multiplets preserve the same number of fermionic zero modes as the perturbative string states, the four graviton scattering in the background of D-brane takes the form

$$\mathcal{A}^{\text{non-pert}}(1,2,3,4) \sim \widehat{K}^2 \times e^{-S_{\text{classical}}}\, f(m^i), \tag{5.1}$$

with the same kinematic factor $\widehat{K}^2$ produced by the fermionic zero modes times a classical weight times a function of the scalars parametrising the vacuum of the vacuum of the theory. This function does not depend on the Mandelstam variables nor the spin of the external states. The spin dependences of the D-branes enter only through the $\widehat{K}^2$ factor [46,47,48].

The appearance of the $t_8 t_8 R^4$ for both the perturbative and non-perturbative amplitudes is a consequence of supersymmetry. There is no other tensorial structure appearing in the supersymmetric invariants which does not involve epsilon ϵ_d tensors.

- *Lifting to eleven dimensions*

The strong coupling limit, $g_s \to \infty$, of the type II theory is the eleven dimension theory [8]. For the non-chiral type IIa theory the connection is done by using the metric [9]

$$ds^2 = G^{(11)}_{MN} dx^M dx^N = \frac{l_{11}^2}{l_s^2 R_{11}}\, g^A_{\mu\nu} dx^\mu dx^\nu + R_{11}^2 l_{11}^2 (dx^{11} - C_\mu dx^\mu)^2, \tag{5.2}$$

the Ramond charges for the D-branes are now part of the eleven-dimension metric. For the chiral type IIb theory, the connection is done through the Aspinwall-Schwarz [11,12] compactification, which involves a flip of the chirality of half (sixteen of) the fermions.

The previously discussed amplitudes can be lifted to an amplitude in eleven dimensions into the form of the kinematic factor times a φ^3 one-loop integral compactified on a circle of radius $R_{11} l_{11}$:[10]

[10] I'm using conventions where the coordinates are dimensionless and the metrics are dimensionfull.

$$\mathscr{A}^{1-\mathrm{loop}}(1,\cdots,n) \sim \widehat{K}^2 \frac{1}{R_{11}} \int d^{10}p \sum_{n\in\mathbb{Z}} \frac{1}{P^2(P+K_1)^2(P+K_1+K_2)^2(P-K_4)^2}, \quad (5.3)$$

with $P=(p, n/R_{11}l_{11})$ and $K_i=(k_i,0)$.

D-brane and massless string states are excitations of the eleven dimensional metric and run in the loops. It is then possible to develop a first quantized approach for the four graviton scattering in eleven dimensions [25,49,50]. The vertex operators for the graviton, and its associated super-partners, are deduced in an usual way, by considering the coupling of a super-particle to the gravity background [50]. Since by dimensional reduction these vertex operators should coincide with the usual superstring vertex operators, the equivalent correlations functions of (4.2) in the context of the world-line formalism are for a super-particle

$$\left\langle \prod_{\alpha=1}^{4} V_g^{(\alpha)}(x_\alpha) \right\rangle_{1-\mathrm{loop}}^{\mathrm{super-particle}} \sim \widehat{K}^2 \times \left\langle \prod_{\alpha=1}^{4} e^{ik^\alpha\cdot X(\alpha)} \right\rangle_{1-\mathrm{loop}}^{\mathrm{super-particle}} .$$

As before a kinematic factor factorizes. The computation can be shown to be free of short distance singularities and contact terms are absents [50]. Again a feature of the supersymmetry invariance of the computation.

- *A full $Sl(2,\mathbb{Z})$ invariant amplitude?*

Through the process of Kaluza–Klein reduction, it is possible to follow the analytic structure of the one-loop diagram. This amplitude splits in a zero external momenta part

$$\begin{aligned}\mathscr{A}^{1-\mathrm{loop}}(K_{1,2,3,4}) \propto & \widehat{K}^2 \times \left(2\zeta(3)\mathcal{V}^{-\frac{3}{2}}E_{\frac{3}{2}}(\Omega,\bar\Omega) + \frac{2\pi^2}{3}\right) + \\ & (s^2+t^2+u^2)\widehat{K}^2 \times \left(\zeta(5)\mathcal{V}^{-\frac{5}{2}}E_{\frac{5}{2}}(\Omega,\bar\Omega) + \frac{4}{\pi^2}\zeta(3)\zeta(4)\mathcal{V}^{-4}\right)\end{aligned}$$

expressed as a modular form of weight 3/2 and a finite part, a non-analytic part $\mathscr{A}^{1-\mathrm{loop,non-ana}}(s,t=u=0)$ behaving like $s^{3/2}$ in eleven, $s\ln(s)$ in ten and $s^{1/2}$ in nine dimensions, and an infinity series of terms with positive power of the radius of compactifications. All the analytic terms have coefficiants invariant under the isometries of the torus of compactification, namely $Sl(2,\mathbb{Z})$ on Ω leaving $\mathcal{V}$ fixed. When going up in dimensions the series of terms sum up to reconstruct the correct non-analytic structure in higher dimensions. Notice that in eleven dimensions, the diagram is only given by the non-analytic cut.

Each coefficients of the series are finite numbers, and contribute to string loop effects. In particular, it is possible to extract one- and two-loop terms [33]

$$
\begin{aligned}
\mathscr{A}_{IIa} &\propto (s^2+t^2+u^2)\widehat{K}^2\left(2\zeta(3)r_A^2 + \qquad 4\zeta(2)e^{2\phi^A}\right)\\
\mathscr{A}_{IIb} &\propto (s^2+t^2+u^2)\widehat{K}^2\left(\qquad 2\zeta(3)r_B^{-4} + \qquad 4\zeta(2)e^{2\phi^B}r_B^{-2}\right)
\end{aligned}
\tag{5.4}
$$

Remarking that these terms do not satisfy the symmetry discussed at the beginning of section 4, the amplitude $\mathscr{A}^{1-\text{loop}}$ cannot be the full answer. Some diagrams have to be added.

• *Limitations of the method*

I explained that for the leading $t_8t_8R^4$ term a first quantized version of the supergravity interaction can be developed. This set up is as close as possible to what is done in string theory and consequently has the same advantages: (a) all the supersymmetric partners running in the loop are automatically summed over and (b) the separation of the spin degrees of freedom and the loop integral appeared immediately. But it has the same disadvantages as the string formalism, the next order which can be guessed as a two-loop diagram is not easily obtained. The presence of an internal propagator makes the analysis of the supersymmetry of the configuration difficult to obtain. For example, the vanishing of the $t_8t_8R^4$ contribution at two loops in string theory has still to be shown explicitly [51]. The two loops diagrams will be constructed using unitarity technics.

6. Unitarity technics in supergravity theories

• *Exact two-particle cuts*

The massless spectrum of $\mathcal{N}_4 = 8$ supergravity consists of a graviton of spin 2, a gravitino of spin 3/2, vectors of spin 1, fermions of spin 1/2, and scalars of spin 0. The spins run over $\Updownarrow = \{0, \pm1/2,\ \pm1, \pm3/2, \pm2\}$. The key relation to unitarity in supergravity is (1,2,3,4 are the external gravitons, the arguments can be extended to any amplitudes related by supersymmetry) [44]

$$
\begin{aligned}
&\sum_{S_1,S_2\in\Updownarrow}\mathcal{M}^{\text{tree}}\left(-\ell_1^{S_1},1,2,\ell_2^{S_2}\right)\mathcal{M}^{\text{tree}}\left(-\ell_2^{S_2},3,4,\ell_1^{S_1}\right)\\
&\qquad = istu\mathcal{M}^{\text{tree}}(1,2,3,4)\times\frac{s^2}{(\ell_1-k_1)^2(\ell_2-k_3)^2(\ell_1-k_2)^2(\ell_2-k_4)^2}\\
&\qquad = -16\widehat{K}^2\times\left(\frac{1}{(\ell_1-k_1)^2}+\frac{1}{(\ell_1-k_2)^2}\right)\left(\frac{1}{(\ell_2-k_3)^2}+\frac{1}{(\ell_2-k_4)^2}\right)
\end{aligned}
\tag{6.1}
$$

which has been reduced using the on-shell relations

$$
\begin{aligned}
-\frac{s}{(\ell_1-k_1)^2(\ell_1-k_2)^2} &= \frac{1}{(\ell_1-k_1)^2}+\frac{1}{(\ell_1-k_2)^2}\\
-\frac{s}{(\ell_2-k_3)^2(\ell_2-k_4)^2} &= \frac{1}{(\ell_2-k_3)^2}+\frac{1}{(\ell_2-k_4)^2}\ .
\end{aligned}
$$

The sum over the spins S_1 and S_2 are performed independently. Since the proof of this identity is independent of the dimension, in this paragraph I will use the words 'helicities' as if everything has been reduced to four dimensions. This relation can be easily proved if one remark that supersymmetric Ward identities [52] impose helicity conservation and the equality of all amplitudes with all particles with the same helicity related by supersymmetry, e.g. $\langle +2,+2|+2,+2\rangle = \langle +3/2,+3/2|+3/2,+3/2\rangle = \sqrt{-s/u}\langle +2,+3/2|+2,+3/2\rangle$.[11] Moreover, fermion chirality conservation for massless fermions implies that $\langle +2,+3/2|-2,-3/2\rangle = 0$. So, choosing in (6.1) the gravitons 1 and 2 with negative helicity, the only non-vanishing amplitude $\mathcal{M}^{\rm tree}(-\ell_1^{S_1},1^-,2^-,\ell_2^{S_2})$ is the one for which the intermediate states, $\ell_{1,2}$ are both gravitons of positive helicity. This relation is valid for any arbitrary combinations of extremal states belonging to the super-graviton multiplet. Hereafter I will drop the helicity indices.

As already stressed before in the right hand side of (6.1) the spin of the incoming particles only enter in the kinematic factor.

Integrating over the momenta ℓ_1 and ℓ_2 restricted to be on-shell and to satisfy momentum conservation at the vertices, $\ell_1 = k_1+k_2+\ell_2$, we obtain

$$
\mathrm{Disc}\int\frac{d^{11}\ell_1}{(2\pi)^{11}}\sum_{S_1,S_2\in\Updownarrow}\frac{i}{\ell_1^2}\mathcal{M}^{\rm tree}(-\ell_1^{S_1},1,2,\ell_2^{S})\frac{i}{\ell_2^2}\mathcal{M}^{\rm tree}(-\ell_2^{S_2},3,4,\ell_1^{S_1}) \qquad (6.2)
$$

which is equal to the expression of the two-particle s-cut for the one-loop φ^3 diagram (4.11) multiplied by the kinematic factor $\widehat{K}^2$

$$
-32\widehat{K}^2\times\mathrm{Disc}\int\frac{d^{11}\ell_1}{(2\pi)^{11}}\frac{i}{\ell_1^2}\frac{1}{(\ell_1-k_1)^2}\frac{i}{\ell_2^2}\frac{1}{(\ell_2-k_3)^2}\ . \qquad (6.3)
$$

This equation is valid for the cut in the s channel. The full amplitude is reconstructed by finding a function which has the correct cuts in all the channels. The function (4.10) satisfies these requirements, as obvious from its expression. It should be stressed again that this equation is a two-particle cut of a scalar field theory since the kinematic factor has been factored in front of the integral.

In a given channel two functions presenting the same discontinuities can still differ by a polynomial or rational function of their variables. But it should be remarked that we only have to consider scalar field theory loop integral without any numerators for which it is possible to show explicitly the exactness of the two-particle cut.

[11] In the language of superstring, since the BRST charge associated with the reparametrization ghosts contains a piece given by the world-sheet supersymmetry charge [53], these super Ward identities are rephrased as picture changing operations. Equation (6.1) is easily derived by considering the low-energy limit of the one-loop superstring amplitude.

Fig. 2: (a) The scalar field theory one-loop diagram contributing to the four-graviton amplitude of compactified eleven-dimensional supergravity. (b) The one-loop $t_8t_8R^4$ counter-term that cancels the cubic ultraviolet divergence.

A direct consequence of formulae (6.2) and (6.3) is that the one-loop diagrams for four points scattering in every dimensions where a supergravity theory exists is given by the kinematic factor $t_8t_8R^4$ multiplied by the four point function for a scalar φ^3 theory represented by the diagram (a) in fig. 2. This confirms the already know result in dimensions lower than ten, but gives another derivation of the result in eleven dimensions. Moreover, this construction makes clear that the $t_8t_8R^4$ terms are given by the super-trace $B_8(M)$ and $B_8(S)$ for massless and short multiplets respectively.

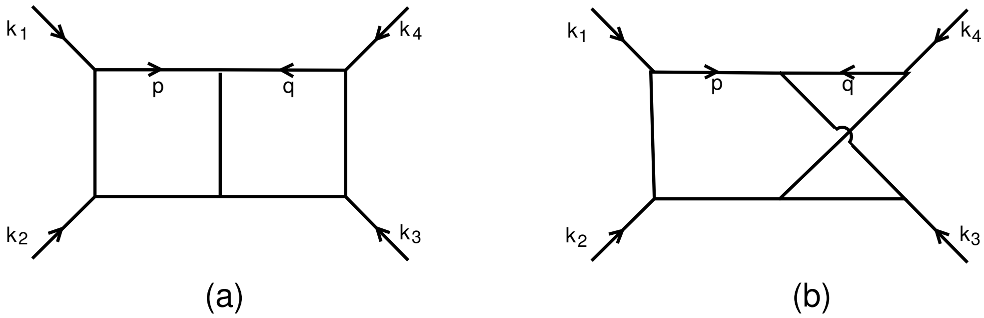

Fig. 3: The 'S-channel' scalar field theory diagrams that contribute to the two-loop four-graviton amplitude of eleven-dimensional supergravity. (a) The (S,T) planar diagram, $I^P(S,T)$; (b) The (S,T) non-planar diagram, $I^{NP}(S,T)$.

Bern *et al.* [44] realized that this procedure can be it iterated as long as the diagram has two-particle cuts. They produced the two loops diagrams of fig. 3. The two-particle s-cut is given by

$$\mathcal{M}^{2-\text{loop}}\big|_{s-\text{cut}} = \text{Disc}\int \frac{d^{11}\ell_1}{(2\pi)^{11}} \sum_{S_1,S_2\in\Updownarrow} \frac{i}{\ell_1^2}\mathcal{M}^{\text{tree}}(-\ell_1^{S_1},1,2,\ell_2^{S_2})\frac{i}{\ell_2^2}\mathcal{M}^{1-\text{loop}}(-\ell_2^{S_2},3,4,\ell_1^{S_1})\,.$$

Inserting the expression for the one-loop amplitude (4.10) and the recursion relation (6.1) we obtain for the two-loop diagrams

$$\mathcal{M}^{2-\text{loop}}(1,2,3,4) \sim \widehat{K}^2 \big[s^2 \left(I^P(s,t) + I^P(s,u) + I^{NP}(s,t) + I^{NP}(s,u)\right) + perms.\big] \quad (6.4)$$

with the following planar

$$I^P(s,t) = \int d^{11}p d^{11}q \frac{1}{p^2(p+k_1)^2(p+k_1+k_2)^2(p+q)^2q^2(q+k_3+k_4)^2(q+k_4)^2} \quad (6.5)$$

and non-planar

$$I^{NP}(s,t) = \int d^{11}p d^{11}q \frac{1}{p^2(p+k_1)^2(p+q)^2(p+q+k_2)^2q^2(q+k_3+k_4)^2(q+k_4)^2} \quad (6.6)$$

scalar φ^3 two loops integrals.

The uniqueness of these expressions has been checked explicitly in [44] by showing that the two loops diagrams have the correct three particle cuts in every dimensions where a supergravity theory exists. This result is valid in every dimensions where a supergravity theory can be defined and where the recursion relation (6.1) holds.

With this construction we do not get any contact term as expected from supersymmetry, despite it is not possible to show this explicitly from explicit computations (with the world-line formalism, for instance) as for the one-loop case.

- *Application: string threshold corrections*

I have constructed one- and two-loop diagrams for the eleven dimensions supergravity theory. These diagrams are plagued by ultra-violet divergences. I regularize them by cutting off the large values of the momenta by Λ. The diagram of fig. 2 divergences as Λ^{11} if considered as a gravity one-loop diagram in eleven dimensions. After having factored out the $t_8t_8R^4$ combination, we are left with a scalar φ^3 diagram with divergences like Λ^3.

The one-loop the cut-off is subtracted by a counter term [25]

$$\delta\mathcal{M}^{1-\text{loop}} \sim c_1 \times \widehat{K}^2 \ , \quad (6.7)$$

In ten dimensions, the natural ultra-violet regulator is the string theory, which gives a finite value to the one-loop $t_8t_8R^4$ term. Using the symmetry between type IIa and type IIb supergravities, induced by the equivalence between the associated superstring theories (see beginning of section 4), one identifies the regulated contribution. The value for this counter-term is fixed to be

$$c_1 = \frac{2\pi^2}{3} - \Lambda^3 l_{11}^3 \ . \quad (6.8)$$

Using the previous cutting technics this counter-term induces a diagram that will contribute at two loops order

$$\delta^{(a)}\mathcal{M}^{2-\text{loop}} \sim c_1 \times (s^2+t^2+u^2)\widehat{K}^2 \times \int \frac{d^{11}p}{(2\pi)^{11}} \frac{1}{p^2(p+k_1)^2(p-k_2)^2} \ . \tag{6.9}$$

This integral is exactly accounted for the cancellation of the sub-leading divergences of the two-loop diagrams of fig. 3. The leading divergence, independent of the number of compactified dimensions, goes as Λ^8 for two loops scalar φ^3 diagrams in eleven-dimensions. To the contrary to the sub-leading divergences this terms would have been regularized by states not present in superstring perturbation theory. The stringy microscopic regularization sets that term to zero. Together the diagrams (6.4) of fig. 3, the counter-term (6.9) (a) of fig. 4 and subtracting the leading divergence with the counter-term (c) of fig. 4, give the finite result

$$\mathcal{M}^{2-\text{loop}}\big|_{\text{Reg}} \sim \left(\mathcal{V}^{-\frac{5}{2}}E_{\frac{5}{2}}(\Omega) + 12\pi^4\zeta(3)\zeta(4)\mathcal{V}^{-4}\right) \times (s^2+t^2+u^2)\widehat{K}^2 \ , \tag{6.10}$$

which once converted in terms of string variables, completes the equations (5.4) into symmetric expressions between type IIa and IIb language.

The previous construction shows that the $D^4\, t_8t_8R^4$ terms are accounted for the contributions from the super-traces $B_{12}(M)$ over massless multiplets and $B_{12}(S)$ short multiplets *only*.

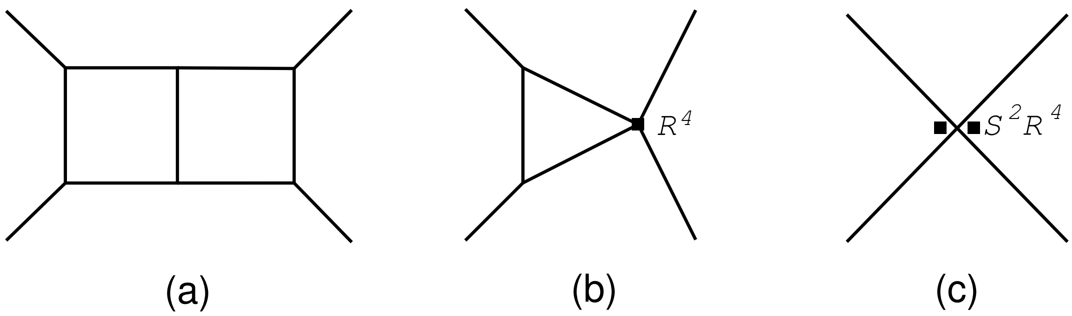

Fig. 4: (a) A scalar field theory two-loop diagram that contributes to four-graviton scattering. (b) One-loop diagrams in which one vertex is the $t_8t_8R^4$ counter-term cancel the sub-divergences of the two-loop diagrams. (c) A two-loop counter-term proportional to $S^2\, t_8t_8R^4$.

• Higher order diagrams

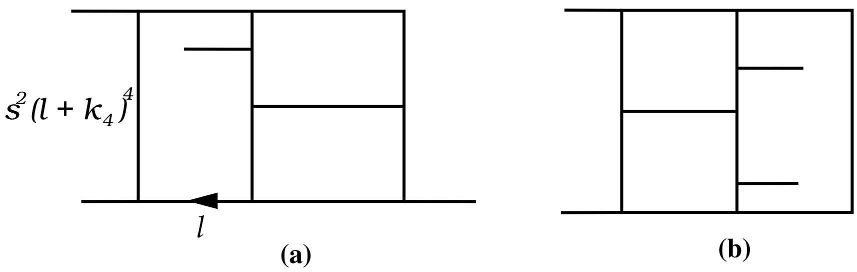

Fig. 5: (a) A three-loop diagram which integral over the loop momenta does not reduces as a scalar two-loop integral. (b) A diagram that contributes to the three-loop amplitude which does not have any two-particle cuts.

Unfortunately at three-loop already exist some two-particles cut constructible diagrams for which the elementary integrals *does not* have interpretations in terms of scalar loop integral (see diagram (a) in fig. 5), and appear some diagrams *not constructible* from two-particle cuts having no such cuts (see diagram (b) in fig. 5). The method can be use with three particles cuts which is less trivial and has not be done yet.

Using dimension analysis arguments, we expected corrections to the $D^4\, t_8 t_8 R^4$ terms from higher-loop diagrams. The classification of the independent tensorial structures at the order k^{12} has not been done neither. Additionally to the $D^4\, t_8 t_8 R^4$ term, R^6-like terms can get contributions. The super-trace over twelve powers of the helicity, B_{12}, gets contributions from the short-multiplet and the intermediate multiplets I_2. It is guessed (despite not yet established) that the terms in the tensorial class of R^6, will get contributions from the super-trace $B_{12}(I_2)$ over the intermediate multiplets I_2 only. To clarify this issue it is necessary to study higher derivative order terms of fig. 4 and the structures of these of fig. 5.

So far the contributions to the eleven-dimensional effective action come from the regularised one-loop R^4 term, since the $D^4 R^4$ term is suppressed by some powers of the volume $\mathcal{V}$ [33]. The next term in the derivative expansion from the two-loop diagram is $D^6 R^4$ and this term can contribute to the eleven-dimensional effective action. In particular like the $D^4 R^4$ which received contributions from the one-loop and the two-loop diagrams this term will receive contributions from the three-loop diagram [44]. At the level of counting only the number of derivative the $D^6 R^4$ is equivalent to R^7 a term that has been argued to contribute in eleven dimensions [54].

7. Instantons computation and the adS/sCFT correspondence

• The conjecture

Maldacena conjectured an equivalence between the weakly coupled type IIb supergravity in the background of the near horizon geometry $adS_5 \times S_5$ of Dthree-brane and the $\mathcal{N}_4 = 4$ SU(N) super Yang–Mills theory at the conformal point ($\beta = 0$) with vanishing Higgs vevs, leaving on the four dimensional boundary of the anti de Sitter space [55].[12] This conjecture has been given a concrete formulation by Witten and others in the form identities between correlation functions computed on the super Yang–Mills side and the one obtained on the supergravity adS-background side with classical sources located at the boundary of the adS space [56,57]. The purpose of this section is to report on the paper by Dorey *et al.* [58], where it has been shown, by evaluating some correlation functions in the k-instantons sector of the super Yang–Mills theory, a complete agreement for the leading order with the supergravity result. This remarkable work is really a strong confirmation of the conjecture. In the course of the argument, it will be shown that the saddle point of the ADHM construction with respect to the rang of the gauge group for fixed instanton number k is given by the zero dimensional matrix model introduced in section 2, giving a strong support for this model as a key model to grasp the physics of the M-theory.

The relevant parameters are the Yang–Mills coupling constant $g_{\mathrm{YM}} = \sqrt{4\pi g_s}$, the theta parameter $\theta = 2\pi C^{(0)}$, the radius of the five-sphere $L^2/\alpha' = \sqrt{g^2_{\mathrm{YM}} N}$, the complex coupling constant $\tau = C^{(0)} + i/g_s = \theta/2\pi + 4i\pi/g^2_{\mathrm{YM}}$.

The limits to consider are weak coupling, $g_s \to 0$, and decoupling of massive string states $\alpha' \to 0$ on the supergravity side, and weak coupling, $g_{\mathrm{YM}} \to 0$ and large-N limit with fixed but large 't Hooft parameter $\lambda = g^2_{\mathrm{YM}} N$. On the super Yang–Mills side an expansion in inverse power of λ is always valid in an instanton background.

$\mathcal{N}_4 = 4$ $SU(N)$ models, have $\beta(g_{\mathrm{YM}}) = 0$ and do not have any chiral anomalies. So the amplitudes are periodic in θ or equivalently are invariant under $\tau \to \tau + 1$. When the Higgs vevs are non-vanishing, the theory is in the Coulomb phase with the gauge group broken to $U(1)^{N-1}$. It is conjectured to have to Montonen-Olive duality symmetry [59,60]. The case relevant for Maldacena's conjecture is the super-conformal case (sCFT) with gauge group $SU(N)$ obtained with vanishing vevs. Super-conformal symmetry alone fixes the value of correlation functions with less or equal to three chiral operators. A constraint is that the correlators should contains enough fermions zero modes to saturate the fluctuations. Here the computation will be done on a sixteen-fermions (gauginos) correlator $G_{16}(x_1, \cdots, x_{16})$ in the large-N limit of the $SU(N)$ $\mathcal{N}_4 = 4$ theory. The number of fermions insertions is fixed by the sixteen exact zero modes of the instantons protected by supersymmetry and super-conformal invariance for vanishing vevs. This correlation function corresponds to the one with sixteen dilatino in supergravity, and is related by supersymmetry transformation to the G_8 and G_4 correlations where the operators are respectively bi-linear and quadri-linear in the fermions.

[12] Having not contributed directly to these developments, this section will be a summary of papers published by other authors.

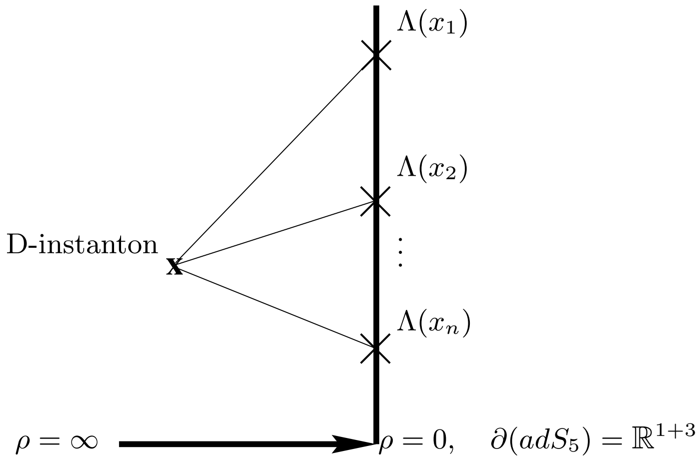

Fig. 6: Diagram showing the geometry of the adS_5 space. A D-instanton located in the bulk space, acts as a non-local effective sixteen-fermions vertex. The classical sources Λ are located on the boundary of the space.

• *The supergravity side*

The low-energy effective theory of N parallel Dthree-branes on the brane world-volume is $\mathcal{N}_4 = 4$ supersymmetric Yang-Mills theory with gauge group $U(N)$ dimensionally reduced to $1+3$ dimensions [9]. The dimensionless four-dimensional gauge coupling is $g_{\rm YM}$. In general the four-dimensional fields which propagate on the brane also have couplings to the ten-dimensional graviton and to other closed string modes. In the limit $\alpha' \to 0$, gravity decouples from the four-dimensional theory. This limit will be considered latter with the string-coupling held fixed and small when turning to the sYM computations. This gives a weakly coupled gauge theory on the Dthree-brane.

Originally Maldacena formulated his conjecture [55], for the large-N decoupling limit, in which the Dthree-brane solution can be reliably described as classical solutions of type IIb supergravity. This leads to the strong coupling limit ($g_{\rm YM}^2 N \gg 1$) of the four-dimensional gauge theory.

The (rescaled) metric around a Dthree-brane in $\mathbb{R}^{1+3}$, has the near horizon geometry (i.e. close to the Dthree-brane), of the anti de Sitter (adS) space $adS_5 \times S_5$

$$ds^2 = \frac{1}{\rho^2}\left(dx^2 + d\rho^2\right) + d^5\Omega \ , \tag{7.1}$$

where $(\rho, x_1, \ldots, x_4)$ parametrise the adS_5 part and $\Omega \in SO(6)$ parametrise the five-sphere S^5. The boundary located $\rho = 0$ is the four-dimensional space $\partial adS_5 = \mathbb{R}^{1+3}$ of the world-volume of the Dthree-brane. For Dthree-brane vacua the dilaton is constant [15]. Moreover, a stack of Dthree-brane on the top of each other does not break anymore supersymmetries and have a near horizon metric still given by (7.1) up to a scaling factor of N. A D-instanton, located in the bulk at the position $(\rho, x_1, \ldots, x_4; \Omega)$ (see fig. 6), contributes to the four graviton amplitude (5.1). This amplitude is proportional to the kinematic factor $t_8t_8R^4$, which is on-shell only a function of the Weyl tensor. The question whether such higher derivative terms can modified the *adS* background as been considered in [61,62]. Kallosh & Rajaraman [62] argue on the basis of the phenomenon of enhance supersymmetry that any higher order corrections cannot alter the exactness of the $adS_5 \times S_5$ geometry.[13]

Since this background is conformally flat the presence of a D-instanton will not modify this background, and one-, two- and three-point graviton scattering correlation functions vanish. The four-point function does not vanish and boundary values for the graviton are sources for the energy-momentum tensor [56,65]. The background (7.1) is maximally supersymmetric with thirty-two real supercharges, the presence of a D-instanton breaks half of the supersymmetries.

The Abelian part of the coordinates in (2.3) are zero-modes that have to be integrated over

$$\int d^{10}x d^{16}\varepsilon\,\sqrt{g} = \int \frac{d^4x d\rho}{\rho^5} d^5\Omega\, d\varepsilon\ .$$

the metric of the *adS* background has been used. The sixteen fermionic zero-modes will be saturated by the fermionic parts of the vertex operators and produce the kinematic factor as discussed before.

This background induces corrections to the effective action for the type IIb supergravity action. I have presented some method to compute the α'^3 and α'^5 corrections; I will now specialized to the case of the α'^3 order corrections. They take the form

$$\mathcal{L} = \frac{1}{\alpha'} \int d^{10}x\sqrt{g}e^{-\phi/2}\left(f^{(0,0)}_{\frac{3}{2}}(\tau,\bar\tau)t_8t_8R^4 + \cdots + f^{(12,-12)}_{\frac{3}{2}}(\tau,\bar\tau)t_{16}\Lambda^{16}\right)\ , \tag{7.2}$$

of non-holomorphic modular functions of weight $(w,-w)$ multiplied by tensor which transforms with the opposite weight under the $SO(2)$ R-symmetry group of the type IIb supergravity.

These modular functions have the following development expressed in term of the sYM variables

[13] The authors of [61] have not discussed the implication of field redefinitions for the metric which allow to generate arbitrary terms involving the Ricci tensor and five-form gauge field dependant terms in the effective. It is believed that the adS/sCFT conjecture is insensitive to such redefinitions being a correspondance between S-matrix elements. To be true this necessitates that $(F^{(5)})^2_{\mu\nu}$ terms are *always* associated with Ricci terms. Higher-derivative order corrections to maximally supersymmetric vacua should be reabsorbable by suitable field redefinitions of the graivtino and the killing spinor. The argument of Kallosh & Rajaraman [62] supposes that the classical superspace structure of the gravity theories in ten dimensions [63] and in eleven dimensions [64] are not altered by higher order derivative terms. I'm not aware of any clear statement concerning this point in the literature.

$$2\zeta(3)e^{-\phi/2}f^{(w,-w)}_{\frac{3}{2}} = \frac{a}{g^4_{\rm YM}}\zeta(3) + b_w + \sum_{k=1}^{\infty}\left(\frac{8\pi^2 k}{g^2_{\rm YM}}\right)^{w+\frac{1}{2}} \mu_2(k)$$
$$\left[e^{-8k\pi^2/g^2_{\rm YM}+i\theta k}\sum_{j=0}^{\infty} c_{-w-4,j-w}\left(\frac{g^2_{\rm YM}}{8\pi^2 k}\right)^{j} + e^{-8k\pi^2/g^2_{\rm YM}-i\theta k}\sum_{j=0}^{\infty} c_{w+4,j+w+8}\left(\frac{g^2_{\rm YM}}{8\pi^2 k}\right)^{j+2w}\right] \tag{7.3}$$

On this expression, we can see that if $\omega \neq 0$ the loop corrections start at a different order for the D-instantons and the anti D-instantons. $\mu_2(k) = \sum_{d|k} d^{-2}$ is a measure factor. In the weak coupling limit $g_{\rm YM} \to 0$ anti D-instanton will dominate. To match the supergravity result and the sYM computation we will have to take a large-N limit and $g_{\rm YM} \to 0$, in which the non-perturbative effects are exponentially suppress. Fortunately, a Fourier transform of the expression with respect to the θ parameter will select the non-perturbative corrections

$$2\zeta(3)e^{-\phi/2}f^{(w,-w)}_{\frac{3}{2}}(\tau,\bar\tau)\Big|_{\rm k-inst} = \left(\frac{8\pi^2 k}{g^2_{\rm YM}}\right)^{n-\frac{7}{2}} \mu_2(k) e^{2ki\pi\tau} \ .$$

For example the sixteen dilatino correlation function will be given by [65]

$$G_{16} = \langle \Lambda(x_1)\cdots\Lambda(x_{16})\rangle \sim \frac{1}{\alpha'} e^{-\phi/2} f^{(12,-12)}_{\frac{3}{2}}(\tau,\bar\tau)\int \frac{d^4x d\rho}{\rho^5}\prod_{i=1}^{16} K^F_{\frac{7}{2}}(X,\rho;x,0) \tag{7.4}$$

where we have introduced the bulk-to-boundary propagator for spin-1/2 field (see fig. 6) [65]. Notice that the τ dependence of the modular function $f_{3/2}$ does not mix with the coordinate dependence on the anti de Sitter space. This means in particular that for arbitrary topological charge k carried by the D-instanton, there is only a single copy of the anti de Sitter space appearing in the volume. This property is a general feature of half-BPS bound states of D-branes of the same kind. Bound states of k D-branes can be imagined as a single D-brane with charge k (or winding number for wrapped branes). This feature will have to be recovered from the instanton computation in the super Yang–Mills side, and will be a non-trivial test of the validity of the conjecture. This property is somewhat against a naive counting of the parameters in multi-instantons sectors.

- *Making Yang–Mills instantons with D-branes*

So far, we have only considered D-instantons in the IIB theory in a flat ten dimensional background and in the absence of other branes.[14] In order to make contact with four-dimensional gauge theory we need to understand how these ideas apply to D-instantons

[14] This sub-section is copied almost verbatim from section IV.2 page 44 et seq. of [58].

in the presence of Dthree-branes. In particular we wish to determine how the D-instanton measure (2.3) is modified by the introduction of N parallel Dthree-branes. Conversely, in the absence of D-instantons, the theory on the four-dimensional world volume of the Dthree-branes is $\mathcal{N}_4 = 4$ $U(N)$ super Yang-Mills theory. Hence a related question is how the D-instantons appear from the point of view of the four-dimensional theory on the Dthree-branes. In fact, the brane configuration considered here is a special case of systems involving configurations of k Dp-branes in the presence of N $D(p+4)$-branes, with all branes parallel. In each of these cases the lower-dimensional brane corresponds to a Yang-Mills instanton in the world-volume gauge theory of the higher [66]. The maximal case $p = 5$, which was first considered (in the context of type I string theory) by Witten [67,68]. The cases with $p < 5$ then follow straightforwardly by dimensional reduction.

We start by considering a theory of k parallel Dfive-branes in isolation. As above, the world-volume theory is obtained by dimensional reduction of ten dimensional $\mathcal{N}_4 = 4$ super Yang–Mills theory with gauge group $U(k)$. The resulting theory in six dimensions has two Weyl supercharges of opposite chirality and is conventionally denoted as $\mathcal{N}_6 = (1,1)$ SYM.[15] After dimensional reduction, the $SO(10)$ Lorentz group of the Euclidean theory in ten dimensions is broken to $H = SO(6) \times SO(4)$. The $SO(6)$ factor is the Lorentz group of the six dimensional theory while the $SO(4)$ is an R-symmetry. The ten dimensional gauge field A_μ splits up into an adjoint scalar a' in the vector representation of $SO(4)$ and a six dimensional gauge field χ_a with $a = 1, \ldots, 6$. Explicitly we set $A_\mu = (a'_n, \chi_a)$ where $n = 0, 1, 2, 3$ is an $SO(4)$ vector index.

In order to describe the fermion content of the theory we will consider the covering group of H, $\bar{H} = SU(4) \times SU(2)_L \times SU(2)_R$. We introduce indices $A = 1, 2, 3, 4$ and $\alpha, \dot{\alpha} = 1, 2$ corresponding to each factor. A ten dimensional Majorana-Weyl spinor can be decomposed into two Weyl spinors of opposite chirality in six dimensions. The corresponding representation of $SO(10)$ decomposes as,

$$\mathbf{16} \to (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}) \tag{7.5}$$

under $\bar{H}$. An explicit decomposition of the ten-dimensional spinor Ψ in terms of the six dimensional spinors, $\mathcal{M}'^A_\alpha$ and $\lambda_{\dot{\alpha}A}$ is given in the Appendix of [58].

The fields $(\chi_a, \mathcal{M}'^A_\alpha, \lambda^{\dot{\alpha}}_A, a'_n)$ form a vector multiplet of $\mathcal{N}_6 = (1,1)$ supersymmetry in six dimensions. In terms of an $\mathcal{N}_6 = (0,1)$ sub-algebra, the $\mathcal{N}_6 = (1,1)$ vector multiplet splits up into an $\mathcal{N} = (0,1)$ vector multiplet containing χ_a and $\lambda^{\dot{\alpha}}_A$ and an adjoint hypermultiplet containing a'_n and $\mathcal{M}'^A_\alpha$. The action of the $\mathcal{N}_6 = (1,1)$ theory is

$$S^{(6)} = \frac{1}{g_6^2} S_{\text{gauge}} + S^{(\text{a})}_{\text{matter}} \ , \tag{7.6}$$

[15] Some convenient facts about six-dimensional supersymmetry are reviewed in [69] (See page 67 in particular).

where

$$S_{\rm gauge}=\int d^6x\,{\rm tr}_k\big(\tfrac{1}{2}F_{ab}^2-\sqrt{2}\pi\lambda_{\dot\alpha A}\left(\bar\Sigma^{AB}_a\mathcal{D}_a\right)\lambda^{\dot\alpha}_B-\tfrac{1}{2}D_{mn}^2\big)\ , \tag{7.7}$$

and

$$S^{\rm (a)}_{\rm matter}=\int d^6x\,{\rm tr}_k\big(\left(\mathcal{D}_a a'_n\right)^2-\sqrt{2}\pi\mathcal{M}'^{\alpha A}\left(\Sigma^a_{AB}\mathcal{D}_a\right)\mathcal{M}'^B_\alpha+i\pi[a'_{\alpha\dot\alpha},\mathcal{M}'^{\alpha A}]\lambda^{\dot\alpha}_A+iD_{mn}[a'_m,a'_n]\big)\ . \tag{7.8}$$

In the above, we have rescaled the fields so that the six-dimensional coupling constant, $g_6^2\sim\alpha'$, only appears in front of the action of the $\mathcal{N}_6=(1,1)$ vector multiplet. I refer to [58] for further details.

Following [66], the next step is to introduce N Dnine-branes of the type IIb theory whose world-volume fills the ten dimensional spacetime.[16] These are analogous to the N Dthree-branes in the $p=-1$ case on which we will eventually focus. The world volume theory of the Dnine-branes in isolation (i.e. in the absence of the Dfive-branes) is simply ten dimensional $U(N)$ supersymmetric Yang-Mills. As explained by Douglas [66], the effective action for this system contains a coupling between the field strength V_{mn} of the world-volume gauge field and the the rank six antisymmetric tensor field $C^{(6)}_{\mu\nu\cdots\rho}$ which comes from the Ramond-Ramond sector of the type IIb theory. The latter field is dual to the three-form field strength which appears in the type IIb supergravity action. This coupling has the form,

$$\int C^{(6)}\wedge V\wedge V, \tag{7.9}$$

where $C^{(6)}$ and V, the gauge field strength, are written as a six-form and a two-form respectively and the integration is over the ten-dimensional world volume of the Dnine-branes. The same six-form field also couples minimally to the Ramond-Ramond charge carried by Dfive-branes. Hence a configuration of the $U(N)$ gauge fields with non-zero second Chern class, $V\wedge V$, acts as a source for Dfive-brane charge. More concretely, if the Dnine-brane gauge field is chosen to be independent of six of the world-volume dimensions and an ordinary Yang-Mills instanton is embedded in the remaining four dimensions, then the resulting configuration has the same charge-density as a single Dfive-brane. Both objects are also BPS saturated and therefore they also have the same tension. Further confirmation of the identification of Dfive-branes on a Dnine-brane as instantons was found in [66] where the gauge-field background due to a type I Dfive-brane was shown to be self-dual via its coupling to the world-volume of a Done-brane probe.

[16] In fact a IIb background with non-vanishing Dnine-brane charge suffers from inconsistencies at the quantum level. This is not relevant here because the Dnine-branes in question are just a starting point for a classical dimensional reduction.

As described above, Dfive-branes appear as BPS saturated instanton configurations on the Dnine-brane which break half of the supersymmetries of the world-volume theory. Conversely, the presence of Dnine-branes also break half the supersymmetries of the Dfive-brane world-volume theory described by the action (7.6). Specifically, the $\mathcal{N}_6 = (1,1)$ supersymmetry of the six-dimensional theory is broken down to the $\mathcal{N}_6 = (0,1)$ sub-algebra described above equation (7.6). To explain how this happens we recall that open strings stretched between branes give rise to fields which propagate on the D-brane world-volume. So far we have only included the adjoint representation fields which arise from strings stretching between pairs of Dfive-branes. As our configuration now includes both Dfive- and Dnine-branes there is the additional possibility of states corresponding to strings with one end on each of the two different types of brane. As the Dfive-brane and Dnine-brane ends of the string carry $U(k)$ and $U(N)$ Chan-Paton indices respectively, the resulting states transform in the $(\mathbf{k}, \mathbf{N})$ representation of $U(k) \times U(N)$.

- *True zero-modes and the moduli space of instantons*

The presence of an instanton in the vacuum breaks some symmetries of the initial theory. As usual the breaking of symmetries will induce a number of zero modes that will characterize the space of configurations for these instantons.

We learnt from equation (7.4) that a configuration of k D-instantons in the background of N Dthree-branes is a single $adS_5 \times S_5$ space with sixteen supersymmetries. This section is devoted to the analysis of the moduli space of instantons on the sYM side.

We consider the super-conformal non-Abelian $SU(N)$ Yang–Mills with a topological number k. For $k = 1$ the configuration of one instanton [70] is characterized by the positions $X_\mu \in \mathbb{R}^{1+3}$, a scale variable $\rho \in \mathbb{R}$ and some global gauge orientation variables $\Omega \in SU(N)/U(1) \times SU(N-1)$, for the bosonic variables. This makes a total of $4 + 1 + 4N - 5 = 4N$ variables. The system having supersymmetry, there are eight zero modes ξ^A_α $(A = 1, \cdots, 4, \alpha = 1, 2)$ and eight $\bar\eta^A_{\dot\alpha}$ $(\dot\alpha = 1, 2)$ for the super-conformal invariance. This makes a total of sixteen (obvious) fermionic zero-modes. The index theorem for the Dirac operator for an (anti-)self-dual gauge field in the sector with topological number k, teaches us that there are $8N - 16$ extra fermionic zero-modes, μ^A_u and $\bar\mu^A_u$ $(u = 1, \cdots, N-2)$, to match the bosonic ones.

The fermion operator in the sYM theory which corresponds to the dilatino inserted in (7.4) is

$$\Lambda^A_\alpha = \frac{1}{g^2_{\rm YM}} \sigma^{mn\beta}_{\ \ \ \alpha} \, {\rm tr}_{\rm N} v_{mn} \lambda^A_\beta \ ,$$

where v_{mn} is the $SU(N)$ field strength projected on it self-dual component by the anti-symmetric product of Dirac matrices σ^{mn}. The restriction to the (anti-)self-dual component selects the leading (anti-)instanton contribution in (7.4). In particular the field strength in the background of one instanton satisfies ${\rm tr} v_{mn} v_{kl} = 32 \mathcal{P}_{mn,kl} K_4(X, \rho; x, 0)$

where K_4 is the bulk-to-boundary propagator and $\mathcal{P}_{mn,kl} = 4^{-1}(\delta_{mk}\delta_{nl} - \delta_{ml}\delta_{nk} + \epsilon_{mnkl})$ is a projector on the self-dual component of the field strength.

The extra zero-modes μ_u^A and $\bar{\mu}_u^A$ correspond to flat directions of the action at the Gaussian level only and are lifted by Yukawa couplings at higher order in the fluctuations,

$$\Delta S = \varepsilon^{ABCD} \left[\sum_{u=1}^{N-2} \bar{\mu}_u^A \mu_u^B\right] \left[\sum_{u=1}^{N-2} \bar{\mu}_v^C \mu_v^D\right] . \tag{7.10}$$

As it is more convenient to deal with quadratic action when a saddle-point has to be obtained, Dorey *et al.* [58] introduced the auxiliary variables χ_{AB} belonging to the **6** of $SO(6) \simeq SU(4)$ subject to the reality condition $\epsilon^{ABCD}\chi_{CD} = 2\chi_{AB}^{\dagger}$ (which can written as a $SO(6)$ vector χ_a later) such that

$$\int d^{4N-8}\bar{\mu} d^{4N-8}\mu \, e^{-\Delta S} = \int d\chi d^{4N-8}\bar{\mu} d^{4N-8}\mu \, e^{-\bar{\chi}_{AB}(\chi^{AB} - \sum_u \bar{\mu}_u^A \mu_u^B)} = \int d^6\chi |\chi|^{4N} \, e^{-|\chi|^2}$$

Splitting χ in a rescaled radial part $N \times r$ and an angular part $\Omega \in S^5$, the functional integral in the one-instanton secteur reads

$$\int d\mu_{1-\mathrm{inst}} \, e^{-S(1-\mathrm{inst})} = C_N g_{\mathrm{YM}}^8 e^{2i\pi\tau} \int d^4X \, d\rho \, d^5\hat{\Omega} \; \rho^{4N-7} \, I_N \prod_{A=1}^{4} d^2\xi^A d^2\bar{\eta}^A$$

where

$$I_N = N^{2N-1} \int_0^\infty \frac{dr}{r^3} e^{2N(\log(r^2) - \rho^2 r^2)} \sim_{N\to\infty} C'_N \, \rho^{2-4N} \; ,$$

which presents a saddle-point for $r = \rho^{-1}$. Thus the one-instanton functional integral takes the form

$$\int d\mu_{1-\mathrm{inst}} \, e^{-S(1-\mathrm{inst})} = C''_N g_{\mathrm{YM}}^8 e^{2i\pi\tau} \int d^4X \, \frac{d\rho}{\rho^5} \, d^5\hat{\Omega} \, \prod_{A=1}^{4} d^2\xi^A d^2\bar{\eta}^A \tag{7.11}$$

of an integral over the adS_5, (X, ρ), and S_5, Ω, variables and their associated fermionic partners ξ and $\bar{\eta}$. The correct measure for the ρ variable and the five-sphere have their origin in the elimination of the extra fermionic zero-modes μ and $\bar{\mu}$ and the identification of the scale of χ with ρ^{-1} at the saddle-point.

I now turn to consider the k-instantons sector. When the topological number is greater than one, things are gettting really complicated, especially when one wants to construct a gauge invariant measure (see [71] for explicit constructions and a list of references). Naively, one imagines that the number of degrees of freedom increases with the topological charge and the moduli space of k-instantons will be k times bigger than for one instanton, $(adS_5 \times S_5)^k$. However, the large-N limit reduces this space of a single copy of $adS_5 \times S_5$, by generating an attractive potential for the instantons. I refer to [58] pages 62 *et seq.* for details. The result is that the perturbations around the maximally degenerate saddle-point solution

$$W^0 = 2\rho^2 \mathbb{I}_{k\times k}, \quad \chi_a = \rho^{-1}\Omega_a \mathbb{I}_{k\times k}, \quad a'_n = -X_n \mathbb{I}_{k\times k}$$

are described by the $SU(k)$ matrix model (2.1). The gauge field is given by the rescaled fluctuations $A_\mu = N^{1/4}(\rho^{-1}\delta a'_n, \rho\delta\chi_a)$ ($\mu = 0, \cdots, 9$). The effective bosonic small-fluctuations action is

$$S_b = -\frac{1}{2}\mathrm{tr}\left(\rho^{-4}[\delta a_n, \delta a_m]^2 + 2[\delta\chi_a, \delta a_n]^2 + \rho^4[\delta\chi_a, \delta\chi_b]^2\right) = -\frac{1}{2N}\mathrm{tr}[A_\mu, A_\nu]^2$$

Likewise, the fluctuations for the supersymmetric and superconformal zero-modes

$$\begin{aligned}\mathcal{M}^A_\alpha &= 4\xi^A_\alpha \mathbb{I}_{k\times k} + 4\delta a_{\alpha\dot\alpha}\bar\eta^{\dot\alpha A} + \delta\mathcal{M}^A_\alpha\\ \zeta^{\dot\alpha A} &= 4\bar\eta^{\dot\alpha A}\mathbb{I}_{k\times k} + \delta\zeta^{\dot\alpha A}\end{aligned}$$

couple to the fluctations of the bosonic field and give

$$S_f(\delta a_n, \delta\chi, \delta\mathcal{M}, \delta\zeta) = \frac{1}{N}\mathrm{tr}\bar\Psi\Gamma_\mu[A_\mu, \Psi]\ .$$

Finally the measure in the k-instantons sectors and in the large-N regime has the form

$$\int d\mu_{\mathrm{k-inst}}\, e^{-S(\mathrm{k-inst})} = C''_N(k) g^8_{\mathrm{YM}} e^{2i\pi k\tau}\ \times \mathcal{Z}_k \times \int d^4X\, \frac{d\rho}{\rho^5}\, d^5\hat\Omega\, \prod_{A=1}^4 d^2\xi^A d^2\bar\eta^A\ . \quad (7.12)$$

Evaluating explicitly the partition function $\mathcal{Z}_k$ [28,29,32,34,72], the prefactor multiplying the integral is given by the kth Fourier coefficient with respect to τ_1 of $f^{(w,-w)}_{3/2}(\tau,\bar\tau)$

$$C''_N(k) g^8_{\mathrm{YM}} e^{2i\pi k\tau}\ \times \mathcal{Z}_k = \left(\frac{8\pi^2 k}{g^2_{\mathrm{YM}}}\right)^{w+\frac{1}{2}} \mu_2(k) e^{2ki\pi\tau}\ .$$

The remaining integral takes care of the supersymmetric $adS_5 \times S_5$ variables as in (7.4).

I would like to conclude this section with some speculative remarks. We saw that in the moduli space of k-instantons collapses to the degenerated space of one-instanton plus the effective model (2.1) for the extra relatives variables in the large-N limit. This behaviour is related to the supersymmetry preservation of the configurations. First, in the context of the Matrix Model, it can be easily argued that region of the moduli space which corresponds to distinct instantons does not contributes to the correlations functions G_{16} (7.4), G_8 and G_4, because of a fermionic selection rule (see section V.(26) of [73]). In particular when computing the partition function $\mathcal{Z}_k$ in the context of two-dimensional compactification, this selection rule implies that only one-component covering of the space of compactification is selected [72].

From the point of vue of bound states of D-branes, this is reinterpreted by saying that there is no differences between k Dzero-branes wrapped one times around a circle and a single Dzero-brane wrapped k-times. I believe this being a property of BPS bound states and will not be shared by non-BPS bound states.

It should be interesting to obtain the measure factor $\mu_4(k) = \sum_{d|k} d^{-4}$ for 1/4th BPS from the same kind of arguments.

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