PoS PROCEEDINGS OF SCIENCE

Next-to-next-to-next-to-leading order pion contributions to hadronic vacuum polarisation

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We report on the analytic evaluation of the hadronic vacuum polarisation in two-flavor chiral perturbation theory to three loops [1] that's relevant for the calculation of the anomalous magnetic moment of the muon and electron. The three-loop amplitude gets contributions from polylogarithms and six elliptic master integrals, all derivable from the single-mass two-point sunset integral via differentiation or integration. Our result is intended to serve as a starting point for phenomenological studies, as well as the computation of finite-volume corrections in lattice QCD.

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1. Introduction

The calculation of the magnetic moment of the electon is one of the crowning achievements of Quantum Field Theory. The measurement and theoretical predictions agree to about one part in a trillion [2]. Moreover, the anomalous contribution magnetic moment of the muon,

$$\vec{\mu}_\mu = \pm g_\mu \frac{e}{2m_\mu}\vec{S}; \qquad a_\mu = \frac{(g-2)_\mu}{2}, \tag{1}$$

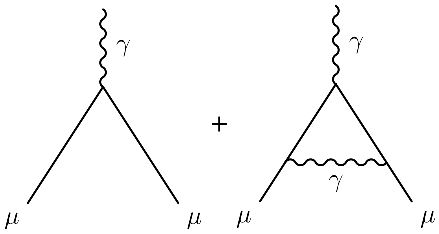
is sensitive to possible new states beyond the Standard Model, which provides an interesting way to probe the new physics.

To compare theory and experiment, highly technical calculations are involved. QED calculations have to be performed at high order, and contributions from hadronic physics become important at such high precision. At the needed precision all three interactions and all standard model particles contribute to the anomalous magnetic moment a_μ, which can be decomposed into a QED a_μ^{QED}, an hadronic $a_\mu^{\mathrm{hadronic}}$ and an electro-weak a_μ^{EW} contribution

$$a_\mu^{\mathrm{SM}} = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{hadronic}} + a_\mu^{\mathrm{EW}}, \tag{2}$$

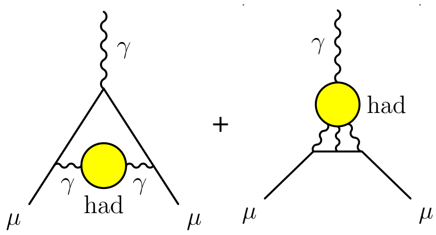
where the terms can be estimated to be of the following orders of magnitude [3]

$$a_\mu^{\mathrm{QED}} =$$



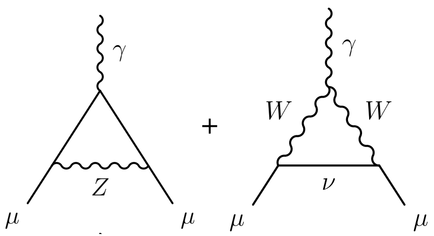
$$= O\left(\frac{\alpha}{2\pi}\right) = O\left(10^{-3} EW\right), \tag{3}$$

$$a_\mu^{\mathrm{hadronic}} =$$



$$= O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) = O\left(10^{-7} EW\right), \tag{4}$$

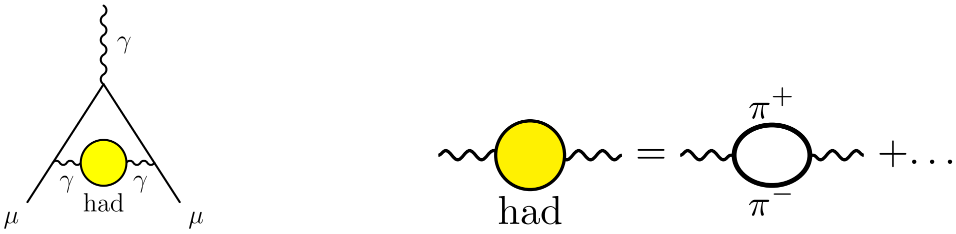
$$a_\mu^{\mathrm{EW}} =$$



$$= O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) = O\left(10^{-7} EW\right). \tag{5}$$

The theoretical uncertainty of the standard model prediction is dominated by the hadronic contributions [4, 5] represented in fig. 1a,

$$a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{QED}} - a_\mu^{\mathrm{EW}} = 718.6(2.2) \times 10^{-10}. \tag{6}$$



(a) The hadronic vaccuum polarisation

(b) The pionic ($\pi^\pm$ and π^0) contributions to the hadronic vaccuum polarisation

Figure 1: Contribution to a_μ^{HPV}.

Therefore, precision in $(g-2)_\mu$ demands theoretical uncertainties below 10^{-10} in the hadronic contribution a_μ^{hadronic}.

At low virtualities, the effect is dominated by non-perturbative QCD effects, in particular pion dynamics. Therefore, from now we will focus on the hadronic contributions at very low energy. That contribution is subleading but small virtualities is where the most uncertainty arise. We work with two-flavor chiral perturbation theory (ChPT) which is a low-energy effective field theory for QCD [6]

$$\mathcal{L}^{\text{QCD}}(q,\bar{q},A) \to \mathcal{L}^{\text{ChPT}}(U,\partial_\mu U,\cdots) = \frac{F_0^2}{4}\text{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \text{higher-order}. \tag{7}$$

The pions enter in the parametrisation of $U := \exp\left(\frac{i\pi^a\sigma^a}{F_0\sqrt{2}}\right)$ where F_0 is the bare pion decay constant, and σ^i the $SU(2)$ generators,

$$\mathcal{L}^{\text{ChPT}} = \mathcal{L}^{\text{ChPT}}_{O(p^2)} + \mathcal{L}^{\text{ChPT}}_{O(p^4)} + \mathcal{L}^{\text{ChPT}}_{O(p^6)} + \mathcal{L}^{\text{ChPT}}_{O(p^8)} + \cdots \tag{8}$$

The EFT lagrangian is ordered in powers of the momentum, and at each order arise new low-energy constants (from UV counter-terms) that are determined by matching physical quantities [7–10].

The basic quantity of interest is the vacuum polarization of fig. 1b,

$$\Pi^{\mu\nu}(q) := i\int d^4x\, e^{iqx}\langle 0\big|T\{j^\mu(x)j^\nu(0)\}\big|0\rangle\,, \tag{9}$$

where $j_\mu(x)$ is the electromagnetic current. The hadronic vacuum polarisation (HVP) consists of the hadronic contributions to $\Pi^{\mu\nu}(q)$, i.e. those that arise from quarks and gluons should be computed in QCD. Here we use two-flavor ChPT to compute the pion contributions to HVP taht we call $\Pi^{\mu\nu}(q^2)$ in the following for simplicity.

Gauge invariance imposes that the longitudinal part of $\Pi^{\mu\nu}(q)$ vanishes and only the transverse part is non-vanishing:

$$\Pi_T(q^2) := \frac{1}{1-d}\left[\frac{\eta_{\mu\nu}q^2 - q_\mu q_\nu}{(q^2)^2}\right]\Pi^{\mu\nu}(q). \tag{10}$$

The primary method for computing the hadronic vacuum polarization is lattice QCD, which suffers from finite volume effects. These effects can be computed in ChPT.

A quick estimation of the contributions to the hadronic vacuum polarisation in perturbation given in fig. 2 shows that to have control of the theoretical uncertainty from finite volume effects at a level matching the experimental uncertainty, the finite volume effect of a three-loop (N^{3}LO) calculation is needed.

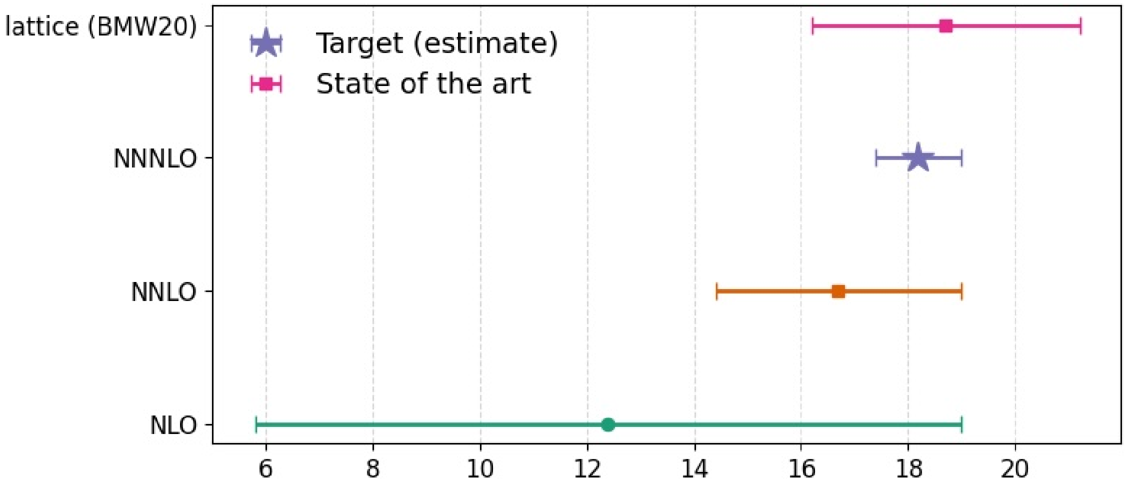


Figure 2: Estimate of the finite-volume effect on $a_\mu^{\mathrm{LO-HVP}}$ in a 6 fm box

2. The HVP at N^{3}LO in ChPT

We work in the two-flavor chiral perturbation theory in the isospin limit, describing an SU(2) triplet of pion fields of mass M_π coupled to an external, non-dynamic photon field A_μ. The structure of the subtracted vacuum polarisation function in an expansion of ChPT up to next-to-next-to-next-leading order (N^{3}LO) may be written as an expansion in powers of $\xi := M_\pi^2/(16\pi^2 F_\pi^2) \approx 0.03$, where F_π is the pion decay constant:

$$16\pi^2 \left(\Pi_T(q^2) - \Pi_T(0)\right) = \bar{\Pi}_T^{\mathrm{NLO}}(t) + \xi\bar{\Pi}_T^{\mathrm{NNLO}}(t) + \xi^2\bar{\Pi}_T^{\mathrm{N^3LO}}(t) + O(\xi^3)\,. \tag{11}$$

Note that on the right-hand side, we write the $\bar{\Pi}$'s as dimensionless functions of $t := q^2/M_\pi^2$. There is no leading order (LO) tree-level contributions. The first two terms are known from earlier work [7, 8, 11–13],

$$\begin{aligned}
\bar{\Pi}_T^{\mathrm{NLO}}(t) &= 2\frac{4-t}{3t}J_{\ominus}^{(1)}(t) + \frac{4}{9}\,, \qquad (12)\\
\bar{\Pi}_T^{\mathrm{NNLO}}(t) &= t\left[\frac{4-t}{3t}J_{\ominus}^{(1)}(t) - \frac{1+3\log(M_\pi^2/\mu^2)}{9}\right]^2 - 4t\left[\frac{4-t}{3t}J_{\ominus}^{(1)}(t) - \frac{1+3\log(M_\pi^2/\mu^2)}{9}\right]l_6^q \\
&\quad - 8tc_{56}^q\,,
\end{aligned}$$

where we have introduced the bubble function

$$J_{\ominus}^{(n)}(t) := \int_0^1 dx\,\log^n\left[1-x(1-x)t\right]\,. \tag{13}$$

The result depends low-energy constants (LEC subtracted HVP). The cut of the amplitude at this order only gives the two pion production channel, e.g. $e^+e^- \to 2\pi$.

The N^{3}LO three-loop order, is new and the topic of the paper [1]. The diagrams contributing to this order are represented in fig. 3, and the result can be expressed as

$$\bar{\Pi}_T^{\mathrm{N^3LO}}(t) = \bar{\Pi}_E(t) + \bar{\Pi}_J(t) + \bar{\Pi}_\zeta(t) + \bar{\Pi}_l(t) + \bar{\Pi}_c(t) + t\,r_1^q + t^2\,r_2^q\,. \tag{14}$$

The contribution $\bar{\Pi}_E(t)$ is given by elliptic functions from the three-loop equal mass sunset integral [14], derivatives of this integral and integrals of this integral. The contribution $\bar{\Pi}_J(t)$ is given by polylogarithms, $\bar{\Pi}_\zeta(t)$ is a rational function containing $\zeta(3)$ and π^2, and finally $\bar{\Pi}_l(t)$, $\bar{\Pi}_c(t)$ contain from the NLO and NNLO low-energy constants.

This is the first evaluation of this contribution at this order in ChPT, and opens the four pions intermediate state, e.g. $e^+e^- \to 4\pi$. An important aspect of the result is the emergence of non-trivial algebraic relations between master integrals that are not generated by integration-by-part identities. These are necessary for renormalisation within the effective theory and ensure that all divergences can be absorbed into the LECs of the chiral Lagrangian.

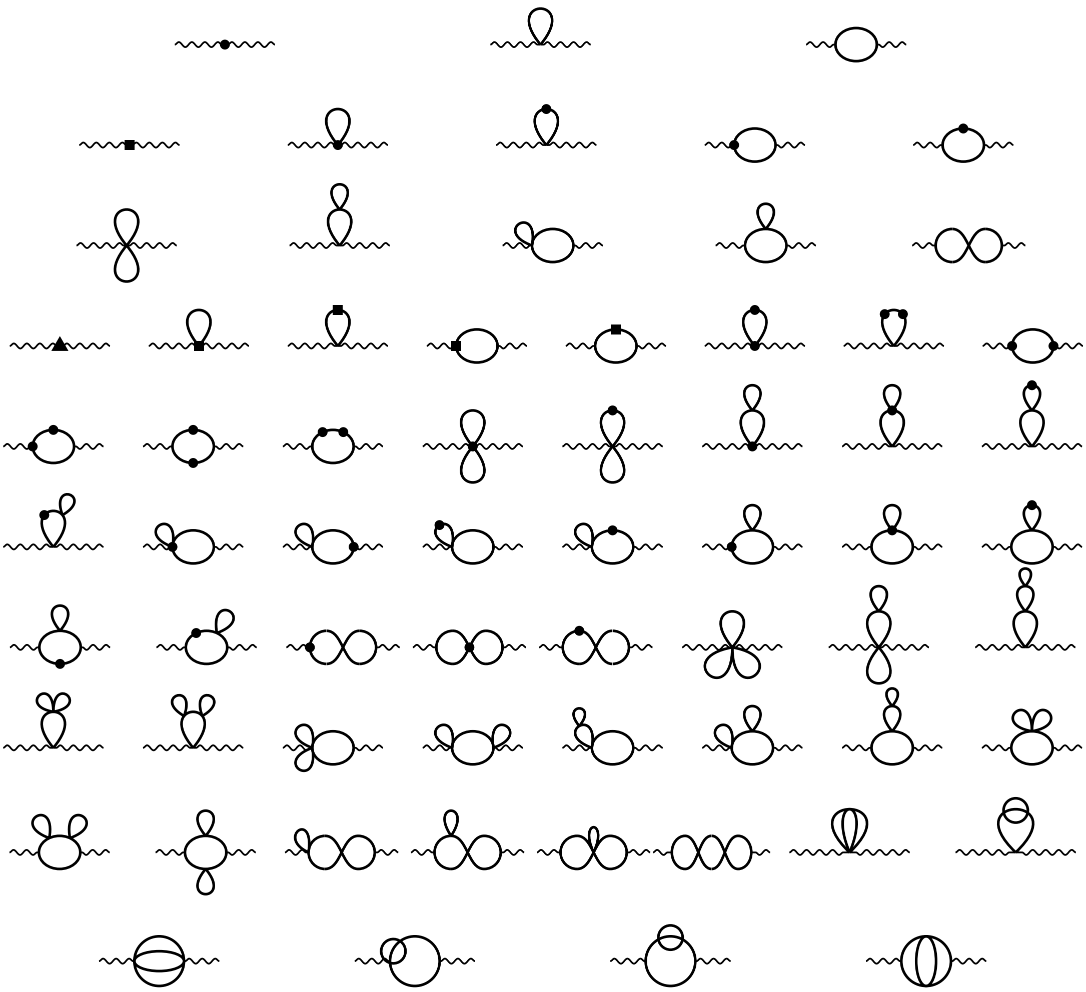


Figure 3: The diagrams contributing to N^{3}LO.

3. Physical interpretation

Traditionally, $\Pi_T(q^2)$ is subtracted at $q^2 = 0$ which corresponds to renormalisazing the electromagnetic coupling in the Thomson limit. We provide the expressions for $\Pi_T(q^2) - \Pi_T(0)$ in [1].

itself is absorbed into a re-definition of LECs (and the electromagnetic coupling renormalisation) so that $\Pi_T(q^2) - \Pi_T(0)$ is finite and scale-independent.

Compared to the lower order, the functional form of the N^3LO contribution (involving elliptic integrals) introduces a more intricate momentum dependence. This complexity matters in dispersive integrals and in lattice interpolation of HVP data, where naive polynomial fits may fail to capture subtle curvature near thresholds. The three-loop contributions correct both the momentum slope and the curvature of the function in a calculable way.

From a phenomenological point of view, the calculation provides several practical insights:

- **Momentum window design:** low-q^2 regions are important for integrals such as the electron and muon $g-2$. The three-loop analytic form indicates that these regions cannot be accurately described by simple polynomials. Fast high-precision evaluations in Python will be released in [15].

- **Finite-volume corrections:** this work is an important first step toward precise estimates of finite-volume effects in lattice calculation of HVP [16].

- **Chiral extrapolation:** the results serve as a refined baseline for extrapolating lattice results obtained at heavier pion masses.

- **Cross-section:** an evaluation of the four-pion production cross-section $e^+e^- \to 4\pi$; the new low energy constants do not contribute so everything needed is known.

- **Error-budgeting:** the numerical magnitude of three-loop effects provides an empirical estimate for the uncertainty associated with truncating the chiral expansion.

4. Conclusion and outlook

From a methodological standpoint, this work also reveals the growing mathematical sophistication of modern effective field theory calculations. The emergence of elliptic integrals, previously seen mainly in multi-loop QED and QCD computations, underscores the deep connections between mathematical physics and phenomenology.

The extension of the HVP calculation to three loops in ChPT opens several avenues:

- Using the analytic results to refine finite-volume correction formulae for lattice HVP simulations.

- Incorporating the elliptic structure into lattice interpolation frameworks and dispersive fits.

- Benchmarking the chiral expansion against lattice data at physical pion masses to test convergence.

This work demonstrates the continuing progress of the theoretical program aiming to match experimental precision in muon $(g-2)_\mu$ and related observables. Effective field theory, lattice QCD and phenomenological methods continue to converge toward a unified, quantitatively reliable description of hadronic effects.

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