Methods for computing observables in classical gravity

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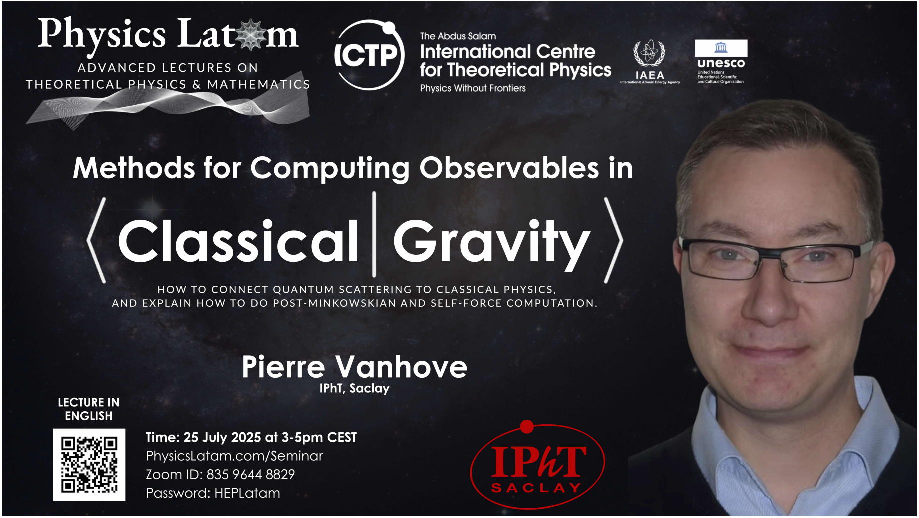
Physics Latam 2025

A method is better than a discovery, because a good method can lead to new results, and much more valuable discoveries.
L. D. Landau

25 july 2025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Methods for Computing Observables in
⟨ Classical | Gravity ⟩
HOW TO CONNECT QUANTUM SCATTERING TO CLASSICAL PHYSICS,
AND EXPLAIN HOW TO DO POST-MINKOWSKIAN AND SELF-FORCE COMPUTATION.
Pierre Vanhove
IPhT, Saclay
LECTURE IN
ENGLISH
Time: 25 July 2025 at 3-5pm CEST
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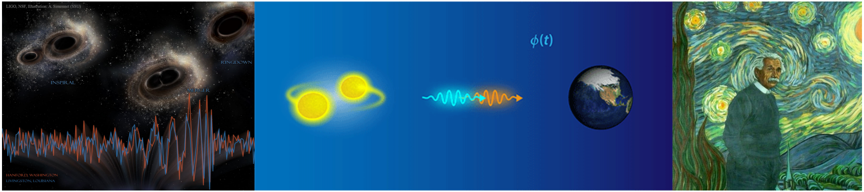
Part I

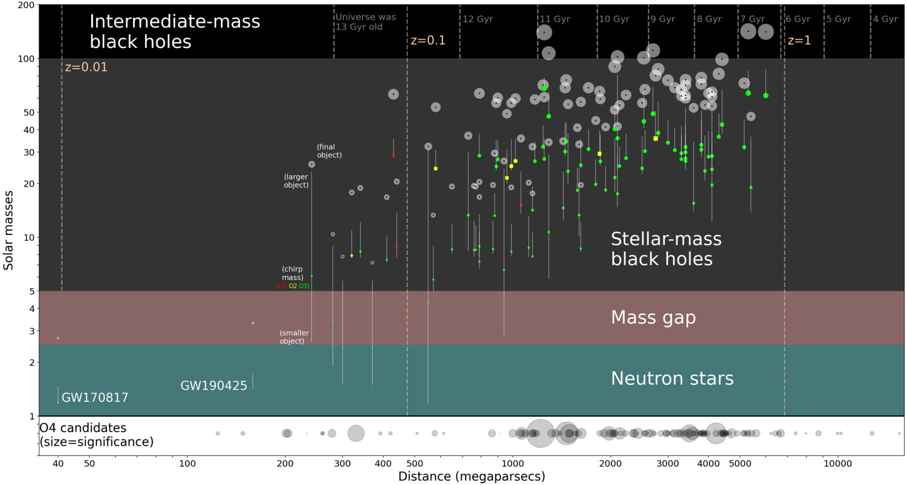
Exploring gravity in our Universe



The measurement of the gravitational wave signal is a formidable window on Einstein theory of gravity and possibly beyond.
Ultimatly this would tell us how good we understand gravity both in the weak and strong coupling regimes.

- The activation of scalar fields
- Gravitational leakage into large extra dimensions
- Higher-derivative corrections from UV completion
- Probing effects on the propagation of gravitational waves
- Test of the strong equivalence principle
- Quantum corrections …

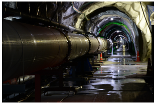
Current O4 events (https://gracedb.ligo.org/)




LIGO Hanford


LIGO Livingston


VIRGO


KAGRA


GEO600

Lower mass gap: GW190711_030756

The LIGO/Virgo collaboration has indeed identified three candidate gravitational wave events from their third observing run (O3) with component masses potentially falling within the lower mass gap, which lies between the heaviest known neutron stars and the lightest black holes.

This is puzzling because there are very little observed candidates black holes of few solar masses above the maximum possible neutron star mass.

The origin of these small black holes is unclear. They may have been created via the merging of binary neutron star systems, rather than stellar collapse.

These events, if confirmed, could provide valuable insights into the formation and evolution of compact objects and the nature of the mass gap

Upper mass gap: GW231123

On 23 November 2023, LIGO has detected the largest binary black hole merger (GW231123) with a black hole of 225 Solar masses for the merger

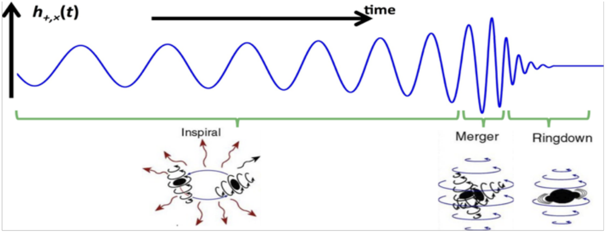
This even particularly puzzling

One or both of the objects involved in the merger may have been in the black hole upper mass gap questioning the current understanding of stellar evolution that predict that black holes with a mass between 60 and 130 Solar masses should extremely rare or forbidden.

Their spin is the highest value observed and close to the external bound

One possible explanation is that these black holes have been created by previous black hole mergers. But they need to have been created in an extremely dense astrophysical environment

A new area of Precision Gravity



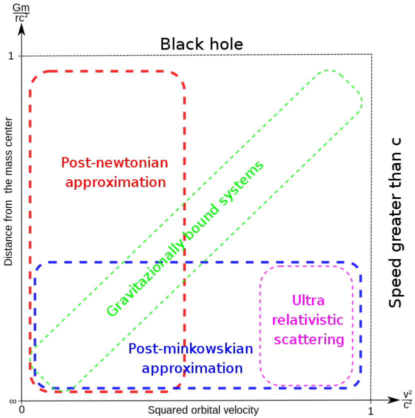
Inspiral-Merger-Ringdown (M. Favata, SXS, K. Thorne)

There are three main analytical approximation methods for describing the two-body dynamics during the inspiral phase

1. the post-Newtonian expansion valid for $(v/c)^2 \simeq G_N(m_1+m_2)/(rc^2) \ll 1$
2. the post-Minkowskian expansion valid for weak field $G_N(m_1+m_2)/(rc^2) \ll 1$ but all values of velocities $\gamma = 1/\sqrt{1-(v/c)^2} \in [1,+\infty[$
3. self-force expansion for small mass ratio $\nu = m/M \ll 1$ for all order in G_N

In this Lecture we will present two new formalisms for the point 2 & 3

Two-body gravitational interactions



The post-Minkowskian expansion gives analytic expression valid from the static case $\gamma = 1$ to the ultra-relativistic Amati-Ciafaloni-Veneziano regime $\gamma \to \infty$. The match is obtained only if one includes gravitational radiation

[Damour; di Vecchia, Heissenberg, Russo, Veneziano; Bjerrum-Bohr, Damgaard, Planté, Vanhove; Driesse et al.]

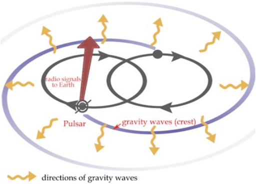
The scattering angle till 5PM (4 loops) and 1SF order $O(m_1 m_2/(m_1+m_2)^2)$

"Emergence of Calabi-Yau manifolds in high-precision black hole scattering" — Mathias Driesse, Gustav Uhre Jakobsen, Albrecht Klemm, Gustav Mogull, Christoph Nega, Jan Plefka, Benjamin Sauer, Johann Usovitsch — arXiv:2411.11846

Part II

From scattering amplitudes to Post-Minkowskian expansion

Post-Minkowskian Hamiltonian

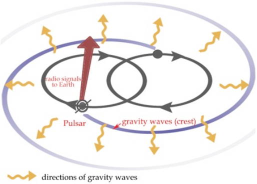


The question of emission of gravitational radiation is a non-relativistic bound state problem

$$\widehat{\mathcal{H}}|\psi\rangle = E|\psi\rangle; \qquad \widehat{\mathcal{H}} = \frac{p^2}{2m} + \widehat{\mathcal{V}}(r); \qquad \widehat{\mathcal{H}}(r) = -G_N \widehat{\frac{1}{r}} + \cdots$$

- For $E > 0$ this is the *scattering regime* we have plane-waves and a *continuous spectrum*
- For $E < 0$ we have the *bound state regime* we have normalized wavefunctions and a *discrete spectrum*

Post-Minkowskian Hamiltonian



A relativistic Hamiltonian for the two-body dynamics in centre-of-mass

$$\widehat{\mathcal{H}}_{\rm PM}(\gamma,r) = \sqrt{\widehat{p}^2+m_1^2}+\sqrt{\widehat{p}^2+m_2^2}+\widehat{\mathcal{V}}_{\rm PM}(\gamma,r),$$

with a relativistic potential organised in a series of Newton's constant G_N

$$\mathcal{V}_{\rm PM}(\gamma,r) = \sum_{L\geq 0}\frac{G_N^{L+1}m_1^2m_2^2}{r^{L+1}}\sum_{r_1+r_2=L} v_{r_1,r_2}(\gamma)m_1^{r_1}m_2^{r_2}$$

which is the general relativity correction to Newton's potential $L=0$

$$\mathcal{V}_1(\gamma,r) = -\frac{G_N}{E_1E_2}\frac{m_1^2m_2^2}{r}(2\gamma^2-1) \qquad \gamma=\frac{p_1\cdot p_2}{m_1m_2}=\frac{1}{\sqrt{1-\frac{\vec{v}^2}{c^2}}}\geq 1$$

Classical physics from quantum loops

THE GENERATION OF GRAVITATIONAL WAVES.
IV. BREMSSTRAHLUNG*†‡

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AND

KIP S. THORNE

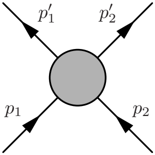
Received 1977 October 21; accepted 1978 February 28

g) The Feynman-Diagram Approach

Any classical problem can be solved quantum-mechanically; and sometimes the quantum solution is easier than the classical. There is an extensive literature on the Feynman-diagram, quantum-mechanical treatment of gravitational bremsstrahlung radiation (e.g., Feynman 1961, 1963; Barker, Gupta, and Kaskas 1969; Barker and Gupta

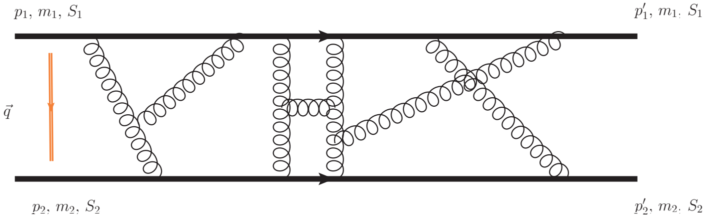
We seek quantum gravity formalism where the classical limit $\hbar \to 0$ gives the general relativity potential

$$\lim_{\hbar\to 0}$$



$$\to \mathcal{M}_L(\gamma, \underline{q}^2) \to \mathcal{V}_{L+1}(p, r)$$

Perturbative gravity



We will be considering the pure gravitational interaction between massive and massless matter of various spin $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G_N} h_{\mu\nu}$

$$\mathcal{S} = \int d^4x \sqrt{-g} \left(-\frac{\mathcal{R}}{16\pi G_N} + \frac{1}{2} \sum_{a=1}^{2} \left(g^{\mu\nu} \partial_\mu \phi_a \partial_\nu \phi_a - m_a^2 \phi_a^2 \right) \right)$$

Evaluating the quantum scattering S-matrix

$$\mathfrak{M}^{\rm GR}(p_1 \cdot p_2, \underline{q}, \hbar) = \langle p_1, p_2 \left| \frac{i}{\hbar} \hat{T} \right| p_1', p_2' \rangle = \sum_{L \geq 0} G_N^{L+1} \mathfrak{M}_L(p_1 \cdot p_2, \underline{q}, \hbar)$$

Gravity effective field theories

We want to develop a formalism that allows to get precise classical post-Minkowskian results but as well is suited for effective field theory extensions of Einstein gravity

We will be working in the context of an effective field theory assuming: [Donoghue]

- Standard QFT (local, unitary, lorentz invariant, ...)
- The low-energy DOF: graviton, usual matter fields
- Standard symmetries: General relativity as we know it

$$\mathcal{S}_{eff} = \mathcal{S}_{eff}^{\text{gravity}} + \mathcal{S}_{eff}^{\text{matter}}$$

$$\mathcal{S}_{eff}^{\text{gravity}} = \frac{1}{16\pi G_N} \int d^4x \, \sqrt{g}\mathcal{R} + \cdots$$

Particles couple to a unique metric $g_{\mu\nu}$: $\mathcal{S}_{eff}^{\text{matter}}(\psi_i, g_{\mu\nu})$

Why use amplitude based methods?

THE GENERATION OF GRAVITATIONAL WAVES.
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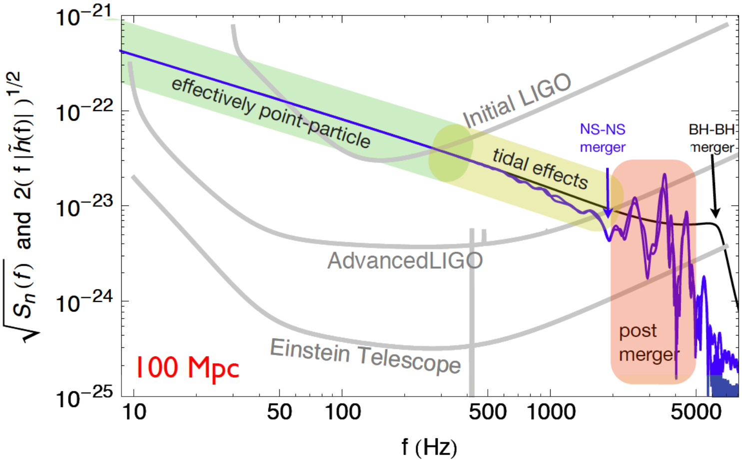
ABSTRACT

This paper attempts a definitive treatment of "classical gravitational bremsstrahlung"—i.e., of the gravitational waves produced when two stars of arbitrary relative mass fly past each other with arbitrary relative velocity v, but with large enough impact parameter that

$$(\text{angle of gravitational deflection of stars' orbits}) \ll (1 - v^2/c^2)^{1/2} .$$

- Recycle knowledge and techniques from particle physics: Efficient amplitude construction : generalised unitarity, double copy, Heavy-Mass Effective Field Theory, algorithms for Feynman integrals computation to high-loop orders (IBP, differential equations, algebraic geometry, ...)
- Clean setup to treat divergencies from IR gravitons
- Recasting the perturbation of classical GR allows to overcome problems of traditional methods: e.g. Kovac-Thorne could not get a closed form formula for the gravitational bremsstrahlung, but amplitude based technique gave it from a unitarity computation at 2-loops [Hermann et al.; Mougiakakos et al.; Jakobsen et al.; di Vecchia et al.]

Point particle approximation



We are using the point particle approximation for describing the binary system (black holes or neutron stars). This is a good approximation till the 6th post-Newtonian order after which finite size effects will be needed.

Many conceptual puzzles, many computational challenges need to be resolved

Equations of motion for compact binary systems in general relativity: Do they depend on the bodies' internal structure at the third post-Newtonian order?

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(Dated: March 6, 2025)

We present and discuss the possibility, derived from work carried out 20 years ago, that the equations of motion for compact binary neutron stars at the third post-Newtonian (3PN) order in general relativity might actually depend on the internal structure of the bodies. These effects involve integrals over the density and internal gravitational potentials of the bodies that are independent of the mass and radius of the bodies, but dependent on their equations of state. These effects could alter the coefficients in the 3PN equations derived using "point mass" methods by as much as 100 percent. They were found in independent calculations done at Washington University using the Direct Integration of the Relaxed Einstein Equations (DIRE) approach, and at the Institut d'Astrophysique de Paris using the Multipolar post-Minkowskian (MPPM) approach. Neither calculation was completed because of the enormous complexity of the algebraic computations and the limitations of software of the day (Maple, Mathematica), and because of an assumption (hope) that the effects would somehow cancel or be removable by some transformation. If these structure-dependent effects are real, but are not incorporated into gravitational waveforms, they could severely impact efforts using next-generation gravitational-wave interferometers to extract information about the equation of state for neutron star matter from gravitational-wave signals from binary neutron star or black hole-neutron star mergers. Conversely, if they exactly cancel or can be absorbed into renormalized masses or shifted positions of each body, this would provide further support for the Strong Equivalence Principle of general relativity.

From amplitudes to post-Minkowskian potential

Setting

$$\widehat{\mathcal{H}}_{\rm PM}(\gamma,r)=\widehat{\mathcal{H}}_0+\widehat{\mathcal{V}}_{\rm PM};\qquad \widehat{\mathcal{H}}_0:=\sqrt{\widehat{p}^2+m_1^2}+\sqrt{\widehat{p}^2+m_2^2}$$

$$\widehat{\mathcal{G}}:=(E_p-\widehat{\mathcal{H}}_{\rm PM}+i\epsilon)^{-1};\qquad \widehat{\mathcal{G}}_0:=(E_p-\widehat{\mathcal{H}}_0+i\epsilon)^{-1};\qquad \widehat{\mathcal{T}}:=\widehat{\mathcal{V}}_{\rm PM}+\widehat{\mathcal{V}}_{\rm PM}\widehat{\mathcal{G}}\widehat{\mathcal{V}}_{\rm PM}$$

The Lippmann-Schwinger equations relate the matrix elements of $\widehat{\mathcal{T}}$ with the ones of $\widehat{\mathcal{V}}$

$$\langle p|\widehat{\mathcal{T}}|p'\rangle=\langle p|\widehat{\mathcal{V}}_{\rm PM}|p'\rangle+\int\frac{d^3\ell}{(2\pi)^3}\frac{\langle p|\widehat{\mathcal{V}}_{\rm PM}|\ell\rangle\langle\ell|\widehat{\mathcal{T}}|p'\rangle}{E_p-E_\ell+i\epsilon}$$

Expanding $\widehat{\mathcal{V}}_{\rm PM}$ in perturbation in G_N we can evaluate

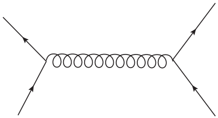
$$\langle p|\widehat{\mathcal{V}}_{\rm PM}|p'\rangle=\langle p|\widehat{\mathcal{T}}|p'\rangle-\int\frac{d^3\ell}{(2\pi)^3}\frac{\langle p|\widehat{\mathcal{T}}|\ell\rangle\langle\ell|\widehat{\mathcal{T}}|p'\rangle}{E_p-E_\ell+i\epsilon}+\cdots$$

When doing a scattering process the scattering amplitude is

$$\frac{i}{\hbar}\langle p|\widehat{\mathcal{T}}|p'\rangle=\mathcal{M}(p,p')$$

we then have a *fully relativitic potential* order by order in perturbation in G_N

One graviton exchange : tree-level (1PM)



$$\mathfrak{M}_0 = -16\pi G_N \hbar \frac{2(p_1\cdot p_2)^2 - m_1^2 m_2^2 - |\hbar \vec{\underline{q}}|^2 (p_1\cdot p_2)]}{|\hbar\vec{\underline{q}}|^2}$$

The $\hbar$ expansion of the tree-level amplitude using $q := \hbar \underline{q}$

$$\mathfrak{M}_0 = \frac{\mathcal{M}_1^{(-1)}(\gamma)}{\hbar|\vec{\underline{q}}|^2} + \hbar 4\pi G_N p_1\cdot p_2$$

The relativistic classical Newtonian potential is obtained by taking the Fourier transform

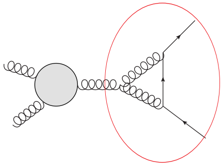
$$V_1(\gamma, r) = \int \frac{d^3\vec{\underline{q}}}{(2\pi)^3}\frac{1}{4E_1E_2}\frac{\mathcal{M}_1^{(-1)}(\gamma)}{|\vec{\underline{q}}|^2} e^{i\vec{\underline{q}}\cdot\vec{r}} = -\frac{G_N m_1^2 m_2^2}{E_1E_2}\frac{2\gamma^2-1}{r}$$

The higher order in $\underline{q}^2$ is the quantum contact interaction of order $\hbar$

Classical physics from loops : One-loop (2PM)

The Klein-Gordon equation reads

$$\left(\Box - \frac{m^2c^2}{\hbar^2}\right)\phi = 0$$



The triangle with a massive leg $p_1^2 = p_2^2 = m^2$ reads

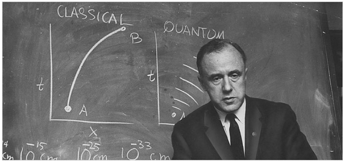
$$\int \frac{G_N^2 \mu^{2\epsilon}\, d^{4-2\epsilon}\ell}{(\ell+p_1)^2(\ell^2 - \frac{m^2c^2}{\hbar^2})(\ell-p_2)^2}\Bigg|_{\text{finite part}}$$

$$\sim \frac{G_N^2}{m^2}\left(\log\left(\frac{q^2}{\mu^2}\right) + \frac{\pi^2 mc}{\hbar\sqrt{q^2}}\right)$$

This one-loop amplitude contains [Iwasaki, Holstein, Donoghue, Bjerrum-Bohr, Vanhove]

- The classical 2nd post-Minkowskian correction G_N^2/r^2 to Newton's potential of order $1/\hbar$
- An infrared quantum correction of order $\hbar^0$

Classical physics from loops : $\hbar$ counting (All PM)



The classical limit $\hbar \to 0$ fixed $\underline{q} \ll m_1, m_2$ of the amplitude [Bjerrum-Bohr, Damgaard, Vanhove, Planté]

$$\mathfrak{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(L+1)}(\gamma)}{\hbar^{L+1}|\underline{q}|^{2+\frac{L(4-D)}{2}}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma)}{\hbar|\underline{q}|^{2-L+\frac{L(4-D)}{2}}} + O(\hbar^0)$$

- A classical contribution of order $1/\hbar$ from all loop orders
- The dimensional regularisation scheme gives a control of the IR divergences from radiation [Di Vecchia, Heissenberg, Russo, Veneziano; Parra-Martinez et al.; Bjerrum-Bohr et al.]
- The computation is explicitly relativistic invariant

Classical physics from loops : $\hbar$ counting

The connection between quantum scattering and classical gravitational physics has forced to rethink the S matrix for dealing with the $\hbar$ expansion

[Damgaard, Planté, Vanhove]

$$\hat{S} = \mathbb{I} + \frac{i}{\hbar}\hat{T} =: \exp\left(\frac{i\hat{N}}{\hbar}\right)$$

doing the Dyson expansion with the conservative and radiation part

$$\hat{T} = G_N \sum_{L\geq 0} G_N^L \hat{T}_L + G_N^{\frac{1}{2}} \sum_{L\geq 0} G_N^L \hat{T}_L^{\rm rad}, \qquad \hat{N} = G_N \sum_{L\geq 0} G_N^L \hat{N}_L + G_N^{\frac{1}{2}} \sum_{L\geq 0} G_N^L \hat{N}_L^{\rm rad}$$

with the completeness relation that includes all the exchange of gravitons

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^D k_1}{(2\pi\hbar)^{D-1}} \delta^+(k_1^2 - m_1^2) \frac{d^D k_2}{(2\pi\hbar)^{D-1}} \delta^+(k_2^2 - m_2^2)$$
$$\times \frac{d^{D-1}\ell_1}{(2\pi\hbar)^{D-1}} \frac{1}{2E_{\ell_1}} \cdots \frac{d^{D-1}\ell_n}{(2\pi\hbar)^{D-1}} \frac{1}{2E_{\ell_n}} \times |k_1, k_2; \ell_1, \ldots, \ell_n\rangle\langle \ell_1, \ldots, \ell_n; k_1, k_2|$$

Exponential formalism

Solving the relation between the operator $\hat{N}$ and $\hat{T}$

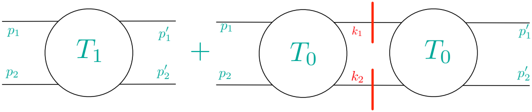
$$\begin{aligned}
\hat{N}_0 &= \hat{T}_0\\
\hat{N}_0^{\rm rad} &= \hat{T}_0^{\rm rad}\\
\hat{N}_1 &= \hat{T}_1 - \frac{i}{2\hbar}\hat{T}_0^2\\
\hat{N}_1^{\rm rad} &= \hat{T}_1^{\rm rad} - \frac{i}{2\hbar}\left(\hat{T}_0\hat{T}_0^{\rm rad} + \hat{T}_0^{\rm rad}\hat{T}_0\right)\\
\hat{N}_2 &= \hat{T}_2 - \frac{i}{2\hbar}(\hat{T}_0^{\rm rad})^2 - \frac{i}{2\hbar}(\hat{T}_0\hat{T}_1 + \hat{T}_1\hat{T}_0) - \frac{1}{3\hbar^2}\hat{T}_0^3
\end{aligned}$$

Taking the vaccum expectation values between the ‘in’ state $|\text{in}\rangle = |p_1, p_2\rangle$ and the ‘out’ state $|\text{out}\rangle = |p_1', p_2'\rangle$

$$\begin{aligned}
\langle p_1, p_2|\hat{N}_0|p_1', p_2'\rangle &= \langle p_1, p_2|\hat{T}_0|p_1', p_2'\rangle\\
\langle p_1, p_2|\hat{N}_0^{\rm rad}|p_1', p_2'\rangle &= 0 \text{ no graviton in the out state}
\end{aligned}$$

Exponential formalism : One-loop

$$\langle p_1,p_2|\hat{N}_1|p_1',p_2'\rangle = \langle p_1,p_2|\hat{T}_1|p_1',p_2'\rangle - \frac{i}{2\hbar}\langle p_1,p_2|\hat{T}_0\mathbb{I}\hat{T}_0|p_1',p_2'\rangle$$



$$\mathbb{I} = \sum_{n=0}^{\infty}\frac{1}{n!}\int \frac{d^Dk_1}{(2\pi\hbar)^{D-1}}\delta^+(k_1^2-m_1^2)\frac{d^Dk_2}{(2\pi\hbar)^{D-1}}\delta^+(k_2^2-m_2^2)$$

$$\times \frac{d^{D-1}\ell_1}{(2\pi\hbar)^{D-1}}\frac{1}{2E_{\ell_1}}\cdots\frac{d^{D-1}\ell_n}{(2\pi\hbar)^{D-1}}\frac{1}{2E_{\ell_n}} \times |k_1,k_2;\ell_1,\dots,\ell_n\rangle\langle\ell_1,\dots,\ell_n;k_1,k_2|$$

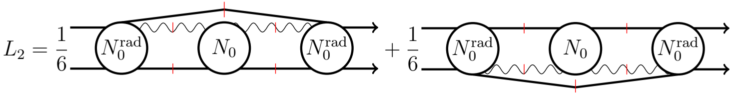
At one-loop the textbook amplitude has the small $\hbar$ expansion

$$\frac{i}{\hbar}\langle p_1,p_2|\hat{T}_1|p_1',p_2'\rangle = \mathfrak{M}_1(\gamma,\underline{q}^2,\hbar) = \frac{\mathcal{M}_L^{(2)}(\gamma)}{\hbar^2|\underline{q}|^{2+\frac{4-D}{2}}} + \frac{\mathcal{M}_L^{(-1)}(\gamma)}{\hbar|\underline{q}|^{2-L+\frac{4-D}{2}}} + O(\hbar^0)$$

The $1/\hbar^2$ piece is subtracted

$$\langle p_1,p_2|\hat{N}_1|p_1',p_2'\rangle = \frac{\mathcal{M}_L^{(-1)}(\gamma)}{|\underline{q}|^{2-L+\frac{4-D}{2}}} + O(\hbar)$$

Velocity cuts



$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^D k_1}{(2\pi\hbar)^{D-1}} \delta^+(k_1^2 - m_1^2) \frac{d^D k_2}{(2\pi\hbar)^{D-1}} \delta^+(k_2^2 - m_2^2)$$

$$\times \frac{d^{D-1}\ell_1}{(2\pi\hbar)^{D-1}} \frac{1}{2E_{\ell_1}} \cdots \frac{d^{D-1}\ell_n}{(2\pi\hbar)^{D-1}} \frac{1}{2E_{\ell_n}} \times |k_1, k_2; \ell_1, \dots, \ell_n\rangle\langle \ell_1, \dots, \ell_n; k_1, k_2|$$

The insertion of complete state induces velocity cuts on the massive line:

$$\delta^+((p_1 + \hbar \underline{q})^2 - m_1^2) = \delta^+(2\hbar p_1 \cdot \underline{q} + \hbar^2 \underline{q}^2) = \frac{\delta(2p_1 \cdot \underline{q})}{\hbar} + O(1/\hbar^2)$$

Exponential formalism

The higher powers of $1/\hbar$ are more singular than the classical contribution, but **are needed for the consistency of the full quantum amplitude and the correct exponentiation of the amplitude**

$$\langle p_1, p_2|\hat N_1|p_1', p_2'\rangle = \langle p_1, p_2|\hat T_1|p_1', p_2'\rangle - \frac{i}{2\hbar}\langle p_1, p_2|\hat T_0\hat T_0|p_1', p_2'\rangle$$

$$\langle p_1, p_2|\hat N_2|p_1', p_2'\rangle = \langle p_1, p_2|\hat T_2|p_1', p_2'\rangle - \frac{i}{2\hbar}\langle p_1, p_2|(\hat T_0^{\rm rad})^2|p_1', p_2'\rangle - \frac{i}{2\hbar}\langle p_1, p_2|\hat T_0\hat T_1+\hat T_1\hat T_0|p_1', p_2'\rangle - \frac{1}{3\hbar^2}\langle p_1, p_2|\hat T_0^3|p_1', p_2'\rangle$$

And they are such that the vaccuum expectation values of the operator $\hat N$ is well defined gives the classical limit [Damgaard, Planté, Vanhove]

$$\langle p_1, p_2|\hat N_L|p_1', p_2'\rangle = N_L^{\rm classical}(p_1, p_2, p_1', p_2') + O(\hbar)$$

The $\hat{N}$ operator upto 1PM and 2PM

$$\hat{S} = \exp\left(\frac{i\hat{N}}{\hbar}\right), \qquad N^{\text{classical}}(\gamma, \underline{q}^2) := \lim_{\hbar\to 0}\langle p_1, p_2|\hat{N}|p_1', p_2'\rangle$$

$$\tilde{N}(\gamma, J) := \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \frac{N(\gamma, \underline{q}^2)}{4m_1 m_2\sqrt{\gamma^2-1}} e^{i\frac{J\cdot\underline{q}}{p_\infty}}; \qquad p_\infty = \frac{m_1 m_2\sqrt{\gamma^2-1}}{\sqrt{m_1^2+m_2^2+2m_1m_2\gamma}}$$

with the result evaluated at tree-level and one-loop using dimensional regularisation $D = 4 - 2\epsilon$

$$\tilde{N}^{1PM}(\gamma, J) = \frac{G_N m_1 m_2(2\gamma^2-1)}{\sqrt{\gamma^2-1}}\Gamma(-\epsilon)J^{2\epsilon}$$

$$\tilde{N}^{2PM}(\gamma, J) = \frac{3\pi G_N^2 m_1^2 m_2^2(m_1+m_2)(5\gamma^2-1)}{4\sqrt{(p_1+p_2)^2}}\frac{1}{J}$$

The 1PM (tree-level) and 2PM (one-loop) contributions are the same as for a test mass in the Schwarzschild black hole of mass $M = m_1 + m_2$.

The $\hat{N}$ operator at 3PM

$$\tilde{N}^{3PM}(\gamma, J) = \frac{G_N^3 m_1^3 m_2^3 \sqrt{\gamma^2-1}}{sJ^2} \times \Bigg(\frac{s(64\gamma^6 - 120\gamma^4 + 60\gamma^2 - 5)}{3(\gamma^2-1)^2} - \frac{4m_1 m_2 \gamma(14\gamma^2+25)}{3} + \frac{4m_1 m_2(3+12\gamma^2-4\gamma^4)\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + \frac{2m_1 m_2(2\gamma^2-1)^2}{\sqrt{\gamma^2-1}} \left(\frac{8-5\gamma^2}{3(\gamma^2-1)} + \frac{\gamma(2\gamma^2-1)\mathrm{arccosh}(\gamma)}{(\gamma^2-1)^{\frac{3}{2}}} \right) \Bigg)$$

At 3PM (two-loop) new phenomena arise

- The conservative part deviates from Schwarzschild as we have contributions which depends (linearly) on the relative mass $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$
- And the important Radiation-reaction terms for the correct high-energy behaviour [Bjerrum-Bohr et al., Para-Martinez et al.; Damour; di Vecchia, Heissenberg, Russo, Veneziano]

Classical observables

The change in an observable $\hat{O}$ is given by the [Kosower, Maybee, O'Connel] expression $\langle\Delta\hat{O}\rangle := \langle \mathrm{out}|\hat{O}|\mathrm{out}\rangle - \langle \mathrm{in}|\hat{O}|\mathrm{in}\rangle$

$$\langle\Delta\hat{O}\rangle(p_1,p_2,r) = \int \frac{d^D(\hbar \underline{q})}{(2\pi)^{D-2}} \delta(2\hbar p_1\cdot \underline{q} - \hbar^2 \underline{q}^2)\delta(2\hbar p_2\cdot \underline{q} + \hbar^2 \underline{q}^2) e^{ir\cdot \underline{q}} \langle p_1', p_2'|\hat{O}|p_1,p_2\rangle$$

which can be expanded using the $\hat{N}$-operator

$$\langle\Delta\hat{O}\rangle = \langle \mathrm{in}|\hat{S}^\dagger \hat{O}\hat{S} - \hat{O}|\mathrm{in}\rangle = \sum_{n\geq 1} \frac{(-i)^n}{\hbar^n n!} \langle \mathrm{in}| \underbrace{[\hat{N},[\hat{N},\dots,[\hat{N},\hat{O}],\dots,]]}_{n \text{ times}} |\mathrm{in}\rangle$$

Because the v.e.v is expressed in terms of nested commutator involving the $\hat{N}$ operator, we have that the $\hbar \to 0$ limit gives directly the classical answer

[Damgaard, Planté, Vanhove]

$$\langle\Delta\hat{O}\rangle = \Delta O^{\mathrm{classical}}(p_1,p_2,r) + O(\hbar)$$

with the exponential representation all superclassical pieces cancel automatically

The radial action

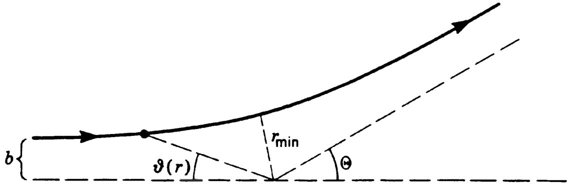
Applying the previous formalism to the momentum kick $\hat{O}^\mu = \hat{P}_1^\mu$ gives in the conservative sector [Bjerrum-Bohr, Damgaard, Planté, Vanhove]

$$\Delta\tilde{P}_1^\mu(\gamma, r)|_{\text{cons}} = -\frac{p_\infty r^\mu}{|r|}\sin\left(-\frac{\partial\tilde{N}(\gamma, J)}{\partial J}\right) + p_\infty^2 L^\mu\left(\cos\left(-\frac{\partial\tilde{N}(\gamma, J)}{\partial J}\right) - 1\right)$$

with the angular momentum

$$L^\mu := \frac{(m_1\gamma + m_2)m_2 p_1^\mu - m_1(m_1 + m_2\gamma)p_2^\mu}{(m_1 m_2)^2(\gamma^2 - 1)}; \qquad p_\infty^2 L^2 = 1$$

The radial action in classical mechanics



In classical mechanics the scattering of a particle against another one (in the frame of one of the particle)the spherically invariant Hamiltonian

$$\mathcal{H} = \frac{p^2}{2m} + V(r); \qquad p^2 = p_r^2 + \frac{J^2}{r^2}; \qquad J^2 = \frac{p_\theta^2 + p_\phi^2}{\sin(\theta)^2}$$

The Hamilton-Jacobi equation for the radial contribution can be rewritten as

$$\frac{1}{2m}\left(\frac{dS(r,\theta)}{dr}\right)^2 + \frac{J^2}{2mr^2} + V(r) = E$$

with the radial action

$$S(r,\theta) := J\theta + N(r,J); \qquad N(r,J) := \int \sqrt{2m(E-V) - J^2/r^2}\, dr$$

The radial action in classical mechanics

The orbit equation is obtained by differantiatng with respect to J and using that $dS(r,\theta)/dJ=0$

$$\theta=-\frac{dN(r,J)}{dJ}=\int_{r_{\rm min}}^\infty \frac{J}{\sqrt{2m(E-V)-J^2/r^2}}\frac{dr}{r^2}$$

The deflection angle is $\Theta=\pi-\theta$. Comparing with the expression for ΔP_1^μ this deduce that in the conservative sector the $\tilde N(\gamma,J)$ is the radial action used by [Landau, Lifshitz; Damour] for computing the scattering angle in classical GR

$$\chi(\gamma,J)=-\frac{\partial\tilde N(\gamma,J)}{\partial J}=\sum_{L=0}^\infty\left(\frac{G_N m_1 m_2}{J}\right)^{L+1}\chi^{(L+1)}_{\rm cons}(\gamma)$$

Part III

Black hole metric

Classical solution from quantum fields

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Quantum Tree Graphs and the Schwarzschild Solution

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I. INTRODUCTION

In an attempt to find quantum corrections to solutions of Einstein's equations, the question naturally arises as to whether the $\hbar \to 0$ limit of the quantum theory correctly reproduces the classical results. Formally, at least, the correspondence between the tree-graph approximation to quantum field theory and the classical solution of the field equations is well known,[1] i.e., the classical field produced by an external source serves as the generating functional for the connected Green's functions in the tree approximation, the closed-loop contributions vanishing in the limit $\hbar \to 0$. The purpose of this paper is to present an explicit calculation of the vacuum expectation value (VEV) of the gravitational field in the presence of a spherically symmetric source and verify, to second order in perturbation theory, that the result is in agreement with the classical Schwarzschild solution of the Einstein equations. This would appear to be the first step towards tackling the much more ambitious program of including the radiative quantum corrections.

In 1973 Duff asked the question about the classical limit of quantum gravity. He showed how to reproduce the Schwarzschild back hole metric from quantum tree graphs to G_N^3 order

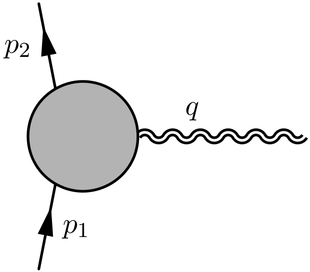
Since then the relation between quantum and classical gravity in amplitude have been rethought with new insights

[Donoghue, Holstein], [Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove], [Kosower, Maybee, O'Connell], [Mougiakakos, Vanhove]

Black hole metric from amplitudes

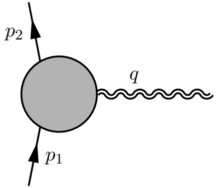
Black hole metric are extracted from the three-point vertex function

- Schwarzschild black hole: Scalar field $S = 0$, mass M
- Reissner-Nordström black hole: Scalar field $S = 0$, charge Q, mass M
- Kerr-Newman black hole: Fermionic field $S = \frac{1}{2}$, charge Q, mass M



$$= -\frac{i\sqrt{32\pi G_N}}{2}\sum_{l\geq 0}\langle T^{(l)\,\mu\nu}(q^2)\rangle\epsilon_{\mu\nu}$$

Black hole metric from amplitudes



$$= -\frac{i\sqrt{32\pi G_N}}{2}\sum_{l\geq 0}\langle T^{(l)\,\mu\nu}(q^2)\rangle \epsilon_{\mu\nu}$$

In the de Donder gauge the metric perturbations are obtained as

$$h^{(l+1)}_{\mu\nu}(\vec x) = -16\pi G_N \int \frac{d^d\vec q}{(2\pi)^d}\frac{e^{i\vec q\cdot\vec x}}{\vec q^2}\left(\langle T^{(l)}_{\mu\nu}\rangle(q^2) - \frac{1}{d-1}\eta_{\mu\nu}\langle T^{(l)}\rangle(q^2)\right)$$

- The classical metric is obtained by using the classical limit of the quantum amplitude $\langle T^{(l)}_{\mu\nu}\rangle^{\text{class.}}(q^2)$ [Bjerrum-Bohr et al.]
- But one can as well include quantum correction to the metric [Donoghue et al.], [Bjerrum-Bohr et al.]

Black hole metric from amplitudes

The scattering amplitudes are done in the *de Donder gauge* coordinate system

$$\eta^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}(g) = \eta^{\mu\nu} g^{\lambda\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right) = 0$$

The Schwarzschild-Tangherlini metric in the de donder coordinate system

$$ds^2 = h_0(r,d)dt^2 - h_1(r,d)d\vec{x}^2 - h_2(r,d)\frac{(\vec{x}\cdot d\vec{x})^2}{\vec{x}^2}$$

$$h_0(r) := 1 - 4\frac{d-2}{d-1}\,\frac{\rho(r,d)}{f(r)^{d-2}},$$

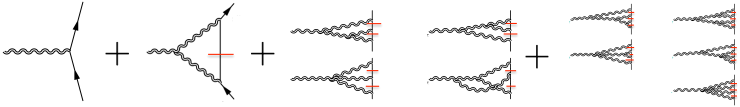
$$h_1(r) := f(r)^2,$$

$$h_2(r) := -f(r)^2 - f(r)^{d-2}\frac{(f(r) + r\frac{df(r)}{dr})^2}{f(r)^{d-2} - 4\frac{d-2}{d-1}\rho(r,d)}.$$

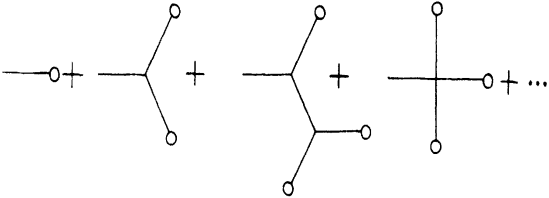
The dimensionless parameter is the post-Minkowskian expansion parameter

$$f(r) = 1 + \sum_{n\geq 1} f_n(r)\rho(r,d)^n; \qquad \rho(r,d) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}\frac{G_N m}{r^{d-2}}$$

Classical contributions from quantum loops



$$\frac{GM}{r} + \left(\frac{GM}{r}\right)^2 + \left(\frac{GM}{r}\right)^3 + \left(\frac{GM}{r}\right)^4$$



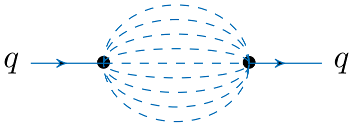
- The tree skeleton graphs are the one computed by Duff
- The velocity cuts are freezing the massive vertices

Classical metric from loops

$$h_{\mu\nu}^{(l+1)}(\vec{q}) = -8\left(c_1^{(l)}(d)(2\delta_\mu^0\delta_\nu^0 - \eta_{\mu\nu}) + c_2^{(l)}(d)\left(2\frac{q_\mu q_\nu}{q^2} + (d-2)\eta_{\mu\nu}\right)\right) \times (\pi G_N m)^{l+1}\frac{J_{(l)}(\vec{q}^2)}{\vec{q}^2}.$$

The metric components in the static limit are given by a single master integral

$$J_{(n)}(\vec{q}^2) =$$



$$= \frac{(\vec{q}^2)^{\frac{n(d-2)}{2}}}{(4\pi)^{\frac{nd}{2}}}\frac{\Gamma\left(n+1-\frac{nd}{2}\right)\Gamma\left(\frac{d-2}{2}\right)^{n+1}}{\Gamma\left(\frac{(n+1)(d-2)}{2}\right)}.$$

Fourier transforming to direct space

$$h_i^{(l+1)}(r,d) = C(d,l)\left(\frac{\rho(r,d)}{4}\right)^{l+1}$$

The de Donder gauge metric in four dimensions

$$h_0^{\mathrm{dD}}(r) = 1 - \frac{2G_N m}{r} + 2\left(\frac{G_N m}{r}\right)^2 + 2\left(\frac{G_N m}{r}\right)^3 + \left(\frac{4}{3}\log\left(\frac{rC_3}{G_N m}\right) - 6\right)\left(\frac{G_N m}{r}\right)^4 + \cdots$$

$$\begin{aligned} h_1^{\mathrm{dD}}(r) = 1 &+ 2\frac{G_N m}{r} + 5\left(\frac{G_N m}{r}\right)^2 + \left(\frac{4}{3}\log\left(\frac{rC_3}{G_N m}\right) + 4\right)\left(\frac{G_N m}{r}\right)^3 \\ &+ \left(-\frac{4}{3}\log\left(\frac{rC_3}{G_N m}\right) + \frac{16}{3}\right)\left(\frac{G_N m}{r}\right)^4 + \left(\frac{64}{15}\log\left(\frac{rC_3}{G_N m}\right) - \frac{26}{75}\right)\left(\frac{G_N m}{r}\right)^5 \\ &+ \left(\frac{4}{9}\log\left(\frac{rC_3}{G_N m}\right)^2 - \frac{24}{5}\log\left(\frac{rC_3}{G_N m}\right) + \frac{298}{75}\right)\left(\frac{G_N m}{r}\right)^6 + \cdots \end{aligned}$$

- The metric is finite but it has powers of $\log(r)$
- The solution has a single constant of integration C_3.
- Determining the generic L loop contribution is difficult

The Schwarzschild metric

P. Fromholz, E. Poisson, C. M. Will explain in their paper "The Schwarzschild metric: It's the coordinates, stupid!" [arXiv:1308.0394] that different choices of coordinate system leads to *very* different form for the Schwarzschild metric.

The amplitude computation is done in the de Donder gauge

$$\eta^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}(g)=0; \qquad g_{\mu\nu}=\eta_{\mu\nu}+\sqrt{32\pi G_N}h_{\mu\nu}$$

which is different from the harmonic gauge condition

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}(g)=0 \Longleftrightarrow \partial_{\mu}\left(\sqrt{-g}g^{\mu\nu}\right)=0$$

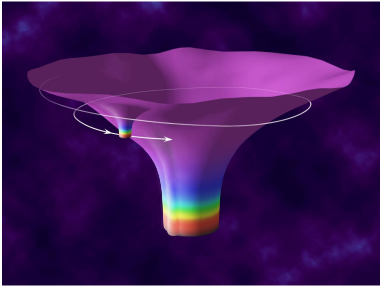
The presence of the $\log\frac{rC_3}{G_N m}$ in the amplitude computation is just a signal that the amplitude computation is done in the wrong gauge, and they can be reabsorbed by a coordinate change

$$(t,\vec{x})\to(t,f(r)\vec{x})$$

Part IV

Self-force

Strong field regime: Self-force expansion



A principal objective of LISA is to investigate the behaviour of general relativity in a strong gravitational field.

The self-force expansion is a expansion in $\nu \sim m/M \ll 1$ for $m_1 = m \ll m_2 = M$ but to all order in G_N

We reorganise the double summation according to the mass-ratio order

$$N(\gamma, J) = \frac{M^3 \nu^2}{|q|^3} \sum_{r\geq 0} \nu^r \sum_{L\geq 2r} (G_N M|q|)^{L+1} c_r^L(\gamma)$$

Self-force expansion

We consider a binary system of a heavy body of mass M and a light body of mass m interacting gravitationally

$$\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_l + \mathcal{S}_H,$$

with the *gothic inverse metric* of the Landau-Lifschits formulation

$$\mathfrak{g}^{ab} = \sqrt{-g}g^{ab} = \eta^{ab} - \sqrt{32\pi G_N}h^{ab}$$

and **the cubic formulation** introduced by [Cheung, Remmen]

$$16\pi G_N \mathcal{S}_{EH} = -\int d^Dx \left(\left(A^a_{bc}A^b_{ad} - \frac{1}{D-1}A^a_{ac}A^b_{bd} \right) \mathfrak{g}^{cd} + A^a_{bc}\partial_a \mathfrak{g}^{bc} \right),$$

and the wordline actions for the light and heavy body

$$\mathcal{S}_l = -\frac{m}{2}\int d\tau_l \left(\frac{\mathfrak{g}^{\mu\nu}v_\mu v_\nu}{\left(\sqrt{-\mathfrak{g}}\right)^{\frac{2}{D-2}}} + 1 \right), \qquad \mathcal{S}_H = -\frac{M}{2}\int d\tau_H \left(\frac{\mathfrak{g}^{\mu\nu}v_{H\mu} v_{H\nu}}{\left(\sqrt{-\mathfrak{g}}\right)^{\frac{2}{D-2}}} + 1 \right).$$

Notice that the auxiliary field A does not couple to the matter field.

Self-force expansion: effective action

Integrating-out the graviton and the auxiliary field we have the effective action

$$e^{i\mathcal{S}_{\text{eff}}[x_l,x_H]} = \int \mathcal{D}h\, \mathcal{D}A\; e^{i\mathcal{S}_{EH}[h,A]+i\mathcal{S}_{GF}[h]+i\mathcal{S}_l[x_l,h]+i\mathcal{S}_H[x_H,h]}.$$

The self-force effective action has an expansion in powers $m/M \ll 1$

$$\mathcal{S}_{\text{eff}} = -\frac{M}{2}\int d\tau_H\; \eta^{\mu\nu} v_{H\mu} v_{H\nu} + M\sum_{n=0}^{\infty}\int d\tau_l\; \left(\frac{m}{M}\right)^{n+1} \mathcal{L}_n[x_l(\tau_l), x_H(\tau_H)]$$

where the leading is the worldline action for the heavy body. We parametrize the trajectory of the light body as

$$x_l^{\mu}(\tau_l) \equiv x^{\mu}(\tau) = \sum_{n=0}^{\infty}\left(\frac{m}{M}\right)^{n} \delta x^{(n)\,\mu}(\tau), \qquad x_H^{\mu}(\tau_H) = u_H^{\mu}\tau_H + \sum_{n=1}^{\infty}\left(\frac{m}{M}\right)^{n} \delta x_H^{(n)\,\mu}(\tau_H),$$

Finally, we should note that we haven't specified the kinematics of the system, therefore the formalism is suitable both for the **bound and the scattering problem**

Deriving exact black-hole metric

The off-shell currents for gravition emission from the heavy source M

$$\sqrt{32\pi G_N}\, h^{(n)}_{\mu\nu}(\mathbf{x}) = \int d^{D-1}x\, e^{i\mathbf{k}\cdot\mathbf{x}}\, J^{(n)}_{\mu\nu}(\mathbf{k}),$$

$$J^{(n)}_{\mu\nu}(\mathbf{k}) = \rho(|\mathbf{k}|, D, n)\left(\chi_1^{(n)}\delta^0_\mu\delta^0_\nu + \chi_2^{(n)}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{\mathbf{k}^2}\right)\right),$$

which leads to the parametrisation of the waveform

$$h^{(n)}_{\mu\nu}(\mathbf{x}) = \rho(r, D)^n\Bigg[\left(\chi_1^{(n)} - \chi_2^{(n)}\right)\delta^0_\mu\delta^0_\nu + \chi_2^{(n)}\frac{1-(n-1)(D-3)}{2-(n-1)(D-3)}\,\delta_{ij} + \chi_2^{(n)}\frac{n(D-3)}{2-(n-1)(D-3)}\,n_\mu n_\nu\Bigg]$$

with a similar parametrisation for the auxiliary field A^a_{bc}.
The order parameter in direct space

$$\int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\mathbf{k}\cdot\mathbf{r}}\rho(|\mathbf{k}|, D, n) = \rho(r, D)^n, \qquad \rho(r, D) = \frac{\Gamma\left(\frac{D-3}{2}\right)}{\pi^{\frac{D-3}{2}}}\frac{G_N m}{r^{D-3}}$$

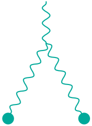
Deriving exact black-hole metric

We compute the emission of the graviton (and the auxiliary field) and identify the form factors.

- At tree-level, i.e. order G_N

$$J^{(1)}_{\mu\nu} =$$  $$= \rho(|\mathbf{k}|, D, 1) 4\delta^0_\mu \delta^0_\nu \Longrightarrow \chi^{(1)}(D) = (4, 0, 0, 0, 0, 0, 0, 0).$$

- one-loop, i.e. order G_N^2

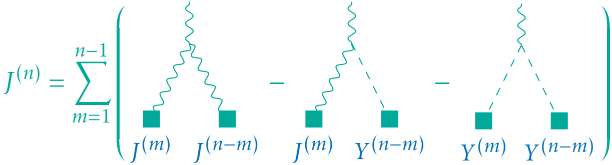
$$J^{(2)}_{\mu\nu} = \frac{1}{2}$$ 

$$J^{(2)}_{\mu\nu} = \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \frac{8\pi^2 \left(G_N m\, \chi^{(1)}_1\right)^2}{(\mathbf{q})^2 (\mathbf{q}-\mathbf{k})^2} \times \left(\left(1 - \frac{(D-3)}{4(D-2)^2}\right) \delta^0_\mu \delta^0_\nu + \frac{(D-3)^2}{4(D-2)^2} \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{|\mathbf{k}|^2}\right) \right)$$

$$= \rho(|\mathbf{k}|, D, 2) \frac{\left(\chi^{(1)}_1\right)^2}{2} \left(\left(1 - \frac{(D-3)}{4(D-2)^2}\right) \delta^0_\mu \delta^0_\nu + \frac{(D-3)^2}{4(D-2)^2} \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{|\mathbf{k}|^2}\right) \right)$$

Self-force expansion: exact metric

At higher the off-sheel current is given by sums of cubic graphs only, therefore presenting a recursive nature



allowing the sum **all loop orders** and lead to a non-perturbative resummation given the **exact** Schwarzschild metric in D dimensions

$$\chi_k^{(n)}(D) = \sum_{i,j=1}^{8} \sum_{m=1}^{n-1} \chi_i^{(m)}(D) \chi_j^{(n-m)}(D) M_k^{ij}(D),$$

Black hole metric

- All integrals are finite and no need to use the non-miminal coupling from EFT approach to cancel divergences from finite size effects
- The result matches exactly the Schwarzschild metric in the "usual" coordinate system, because the computation is done in the harmonic gauge

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0 \qquad \Longleftrightarrow \qquad \partial_\mu \mathfrak{g}^{\mu\nu} = 0$$

- This gives an all order in G_N derivation of the Schwarzschild-Tangherlini metric in general dimensions D.

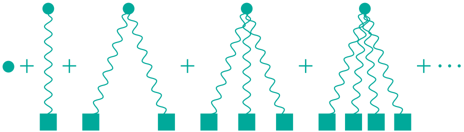
This is a first time infinite resummation to all order in pure gravity.

Self-force expansion: geodesic (0SF)

The geodesic equation for the light mass m is obtained by a **double infinite resummation**:

1. loop for generating the Schwarzschild metric from the heavy mass M
2. the loops for the graviton coupling between the background generated by the heavy mass and the light mass

$$\mathcal{L}_0[x^\mu(\tau), u_H^\mu \tau_H] =$$



This gives a totally consistent self-force formalism without any need to introduce an external metric.

Part V

Effective one-body (EOB) formalism

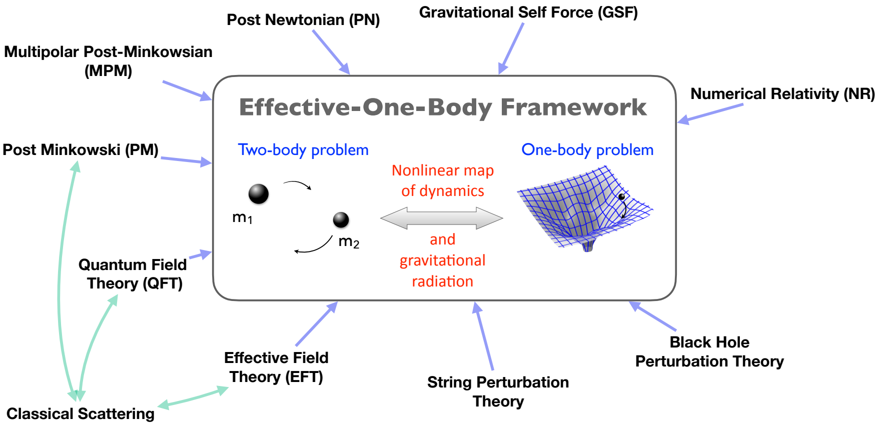
From Scattering amplitudes to EOB

The Effective One Body formalism introduced by Buonanno and Damour in 1998 is an analytical approach to the gravitational two-body problem in general relativity aims to describe all different phases of the two-body dynamics, by building an effective metric

The formalism was historically developed within the post-Newtonian but it has been extended to the post-Minkowskian formalism.

The main idea is to build an *effective metric* that resums the finite number of contributions from perturbations with a specific matching with numerical GR when perturation is not valid anymore

From Scattering amplitudes to EOB



LISA waveform white paper

Waveform Modelling for the Laser Interferometer Space Antenna—2311.01300

From Scattering amplitudes to EOB

One seeks an effective energy $\mathcal{E}_{\rm eff}$, and effective metric $A^{\rm eff}(r)$ and $B^{\rm eff}(r)$

$$ds^2_{\rm eff} = A^{\rm eff}(r)dt^2 - B^{\rm eff}(t)\Big(dr^2 + r^2(d\theta^2 + \sin(\theta)^2 d\varphi^2\Big); \quad \mathcal{H}_{\rm eff} = M\sqrt{1 + 2\nu\left(\frac{\mathcal{E}_{\rm eff}}{\mu} - 1\right.$$

In this formalism one can compute the scattering angle in terms of the effective metric and energy

$$\frac{\chi}{2} = b\int_{r_m}^{\infty} \frac{dr}{r^2} \frac{1}{\sqrt{\frac{B^{\rm eff}(r)}{A^{\rm eff}(r)}\frac{\mathcal{E}^2_{\rm eff}}{p^2_{\rm eff}} - \frac{b^2}{r^2} - \frac{B^{\rm eff}(r)\mu^2}{p^2_{\rm eff}}}} - \frac{\pi}{2}.$$

with

$$p_{\rm eff} = \frac{p_\infty E}{m_1 + m_2}$$

From Scattering amplitudes to EOB

From the amplitude computation we have the scattering angle

$$\frac{\chi}{2} = b\int_{\hat{r}_m}^{\infty} \frac{dr}{r^2}\frac{1}{\sqrt{1-\frac{b^2}{r^2}-\frac{V_{\rm PM}(r,E)}{p_\infty^2}}} - \frac{\pi}{2}.$$

with the four dimensional post-minkowskian effective potential

$$V_{\rm PM}(r,E) = -\sum_{n=1}^{\infty} f_n\left(\frac{G_N M}{r}\right)^n.$$

By matching the scattering angle computed from the effective EOB metric one obtains that the PM expanded potential is related to the metric coefficients by

[Damgaard; Vanhove]

$$1-\frac{V_{PM}(r,E)}{p_\infty^2} = \frac{B(r)}{\gamma^2-1}\left(\frac{\gamma^2}{A(r)}-1\right)$$

in isotropic coordinate the metric can be parametrised by a single function

$$A(r) = \left(\frac{1-h(r)}{1+h(r)}\right)^2; \quad B(t) = (1+h(r))^4; \quad h(r) = \sum_{n\geq 1} h_n (G_N M/r)^n$$

which can be solved iteratively powers of G_N

Post-Minkowskian expansion vs Numerical GR

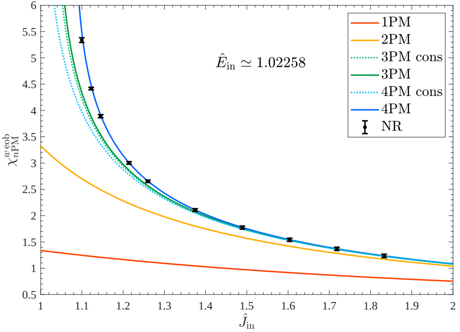


fig 3 of Thibault Damour and Piero Rettegno [arXiv:2211.01399]

Perturbative methods from scattering amplitude can be apply directly leading to new analytic results up to the 4th post-Minkowskian order (3-loop)

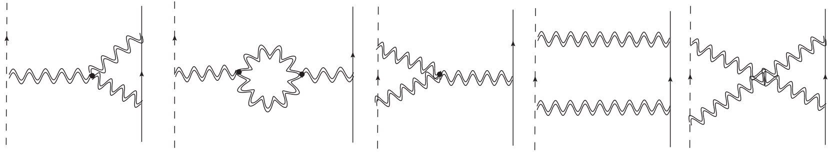
[Bern et al.; Bjerrum-Bohr et al.; Damgaard et al.; Plefka et al.; Porto et al.; Driesse et al.]

with excellent results when compared to numerical general relativity if one includes radiation reaction [Damour, Rettegno]

Part VI

Beyond Einstein theory of Gravity

Quantum correction to the bending angle



Scattering of a massless particle of spin S against a massive scalar one obtains from the one-loop computation [Bjerrum-Bohr, Donoghue, Holstein, Planté, Vanhove]

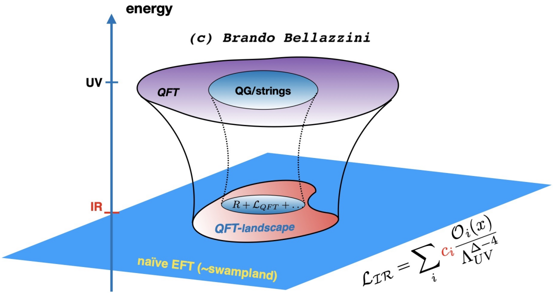
$$\theta_S \simeq \frac{4GM}{b} + \frac{15}{4}\frac{G^2M^2\pi}{b^2} + \frac{8bu^S + 9 - 48\log\frac{b}{2b_0}}{\pi}\frac{G^2\hbar M}{b^3}.$$

The difference between of bending angle shows an intriguing dependence on the spin induced by quantum effects

$$\theta_\gamma - \theta_\varphi = \frac{8(bu^\gamma - bu^\varphi)}{\pi}\frac{G^2\hbar M}{b^3}.$$

Constraining beyond Einstein gravity

This provides way of constraining possible corrections to Einstein's gravity



- ☞ Quantum gravity correction to the star light bending
- ☞ Quantum gravity corrections effects to the metric of black hole solutions
- ☞ Quantum contributions to the causal cone …

[Bellazzini, Isabella, Riva; Madalcena, Zhiboedov; Caron-Huot, Para-Martinez, ...]

The multi-messager detections improves the constraints on various modified gravity models [Gubitosi, Piazza, Vernizzi]

- ✓ the gravitational waves propagates $|c_{GW} - c| < 10^{-15} c$
- ✓ local measurements constraining models for dark energy

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